Analytical Approach to Construction of Tetrahedral Satellite Formation

Sergey Shestakov,* Mikhail Ovchinnikov,† and Yaroslav Mashtakov‡
Russian Academy of Sciences, 125047, Moscow, Russia

https://doi.org/10.2514/1.G003913

The paper discusses the problem of maintenance of the tetrahedral configuration of four satellites. They are assumed to move passively along near-circular orbits and, in linear approximation, their relative motion is finite. The main goal is to define such initial conditions of satellite motion that allow the tetrahedron to preserve its volume and shape. The classification of the obtained solution is presented. General expressions for the initial parameters determination, as well as solution families for some special cases, are obtained. A numerical study that includes J_2 perturbation is conducted for different types of orbits.

I. Introduction

UTILIZATION of several spacecraft in a single mission has a number of benefits in comparison with a single spacecraft mission. Such missions are more redundant because the equipment is divided into several spacecraft, which allows the avoidance of a total loss in case of one of the spacecraft experiences failure. But the most important advantage is the new possibilities that they provide. First of all, it is the coverage: when a bunch of satellites can provide image of the whole Earth surface every day (Planet Labs constellation) or allow us to determine our position at every place on Earth (Galileo, GLObal NAvigation Satellite System, or the Global Positioning System).

However, not only highly distributed formations are used in applications; close ones, when the distance between satellites is about several kilometers, can provide useful scientific data. There are successful missions when satellites have to fly following a given geometry in an orbital reference system, for example, Hyperspectral PRecursor of the Application Mission, gravity recovery and climate experiment, etc. [1–4]. The present paper is focused on tetrahedral satellite formation; more accurately, the main goal is to obtain a set of initial conditions for a group of four satellites so that they form a tetrahedron meeting certain criteria. This concept is especially suitable to the scientific missions with the goal of geomagnetic field exploration. Because single spacecraft missions are incapable of distinguishing temporal and spatial variations of the geomagnetic field, the only continuous source of multipoint measurements is the three-dimensional cell of satellites, i.e., the group of four [5,6]. In recent years, significant effort has been devoted to this problem; several missions were proposed, investigated thoroughly, and successfully launched: the examples include the Auroral Lites mission [7], Cluster II [8], and the Magnetospheric Multiscale mission.

The Magnetospheric Multiscale (MMS) mission should be described here more thoroughly because the success of the mission was one of the motivators for the authors to develop research. The MMS mission is an example of the most deeply developed satellite formation mission: both from engineering and scientific points of view. The goal of MMS is to study small-scale processes that occur in the magnetosphere, as well as the structure of its different regions. To acquire the necessary data, measurements must be taken by a three-dimensional nondegenerate formation of satellites. To achieve, it the particular criteria were developed for a mission construction: the tetrahedron formed by the formation should be as regular as possible while orbiting throughout the region of interest near the apogee of the orbit [9].

The definition of “being as regular as possible” was defined in a strict mathematical way: the scalar quality factor was defined for a generic tetrahedron [10,11]. This factor could be defined several possible ways [12]: the MMS mission uses the ratio of volume of a given tetrahedron and the volume of a regular tetrahedron with the same average side length as the given one [13–18]. The proposed method produces a quality factor that lies between zero and unity and is equal to zero (one) only for a degenerate (respectively, regular) tetrahedron.

Moreover, to maintain a tetrahedron with a side length laying in a given interval, the quality factor is multiplied by a spline function, which nullifies the quality factor for a tetrahedron with an undesirable size. To maintain a regular tetrahedron throughout the whole region of interest, the integral of the quality factor was numerically approximated and then maximized [19,20].

A quite challenging task for this mission design was to appropriately choose initial orbit parameters for each satellite. Because the satellite is subject to different external disturbances, such as the J_2 perturbation and solar radiation pressure, inappropriate initial conditions might lead to a fast degradation of the tetrahedron. There are several papers that investigated the problem of the design and control of formation using a solar sail. A paper [21] proposed solar sail formation flying for exploring the geomagnetic tail. This possibility and the problem of solar sail formation design were explored in detail using analytical and numerical optimizations for two-, three-, and four-craft formations [22,23]. The GeoSail mission concept has been designed to achieve solar sail propulsion control while providing scientific data along the Earth’s geomagnetic tail [24]. For close formation flying, so-called J_2-invariant orbits [25,26] might be used, when the distance between two satellites remains the same even under the influence of the J_2 perturbation. However, their utilization imposes strict restrictions on orbit parameters; therefore, it might be impossible to construct the tetrahedron.

This paper complements the existing body of work in the following way: we want to find solutions for a problem similar to the one of the MMS mission but on the near-circular orbits. It might be necessary if we want to study the geomagnetic field (or for any other scientific application that requires simultaneous and distributed measurements) [27] at low Earth orbit. In addition, we suppose that measurements should be provided for the whole duration of the mission; thus, the tetrahedron must always “be as good as possible.” We propose an index describing the distortion of the tetrahedron that is similar but not quite equal to the one used in the MMS mission; we also provide a solution to the initial value problem that keeps the geometry of the formation approximately fixed. It should be mentioned that, for analytical investigation, we will use a simple Hill–Clohessy–Wiltshire

Received 20 June 2018; revision received 13 April 2019; accepted for publication 29 August 2019; published online 30 September 2019.

*Junior Researcher, Attitude Control and Guidance Division, Keldysh Institute of Applied Mathematics; shestakov.sa@gmail.com.

†Head of Division, Attitude Control and Guidance Division, Keldysh Institute of Applied Mathematics; ovchinni@keldysh.ru. Senior Member AIAA.

‡Junior Researcher, Attitude Control and Guidance Division, Keldysh Institute of Applied Mathematics; yarmashtakov@gmail.com.

model. Although the obtained results could not be implemented directly for the full model of motion, they might be used as a starting point for the numerical optimization problem.

II. Problem Statement and Motion Model

We state the problem as follows:

1) Four satellites orbit passively on near-circular orbits, forming a tetrahedron.

2) The main goal is to find the initial parameters for the satellite motion so that the tetrahedron does not change its size and shape over time (the mathematical equivalent of size and shape preservation will be introduced later).

3) Additionally, we will suppose that measurements must be provided for the whole duration of the mission, and so the tetrahedron should never reach zero volume.

We use the following right-handed Cartesian reference frames:

1) The first reference frame is the inertial reference frame (IRF). Its center $O_\oplus$ is at the Earth center of mass, the axis $O_\oplus Z$ is directed along the Earth axis of rotation, and the axis $O_\oplus X$ is directed to the vernal equinox corresponding to the epoch J2000.

2) The second reference frame is the orbital reference frame (ORF). Its center O is at the one of satellites, the axis Ox is directed along the radius vector of the point O away from the Earth, and the axis Oz is normal to the orbital plane and is directed along the orbital momentum.

A. Motion Model

We use the following assumptions:

1) One of the satellites moves along a circular orbit.

2) The other three move along near-circular orbits.

3) There is no first-order relative drift between the satellites, and so they never fly away from each other in linear model of motion.

The center of ORF is located in the satellite moving along the circular orbit. Without loss of generality, we refer to this satellite as "the fourth." Its motion in the ORF is described by

$$\boldsymbol{r}_4(t) = \langle x_4(t), y_4(t), z_4(t)\rangle = \langle 0, 0, 0\rangle$$

The first and the second assumptions allow us to describe the relative motion of other satellites using the linearized Clohessy–Wiltshire equations:

$$\ddot{x} - 2n\dot{y} - 3n^2x = 0,$$
$$\ddot{y} + 2n\dot{x} = 0,$$
$$\ddot{z} + n^2 z = 0,$$

where $n = \sqrt{\mu/\rho^3}$ is the mean motion, μ is the Earth gravitational parameter, and ρ is the radius of circular orbit. The third assumption guarantees periodic motion of each satellite in the ORF, and so the tetrahedron size is bounded over time. This motion (in the linear model) is then described by the following equations:

$$\begin{aligned} x_i(t) &= A_i \sin\nu + B_i\cos\nu, \\ y_i(t) &= 2A_i\cos\nu - 2B_i\sin\nu + C_i, \\ z_i(t) &= D_i\sin\nu + E_i\cos\nu \end{aligned} \tag{1}$$

where $\nu = nt$. Here, A_i, B_i, C_i, D_i, and E_i are constants depending on the initial values of motion; and index i attains values of one, two, and three. The motion of the fourth satellite is described by the same set of equations with all the constants being equal to zero.

B. Volume Conservation

We now derive the conditions for the tetrahedron to preserve size and shape. The natural measure for the size of the tetrahedron is volume $\mathbb{V}$. In the ORF, the volume has the form

$$\mathbb{V} = \frac{1}{6}\det\|\boldsymbol{r}_1 - \boldsymbol{r}_4, \boldsymbol{r}_2 - \boldsymbol{r}_4, \boldsymbol{r}_3 - \boldsymbol{r}_4\| = \frac{1}{6}\det\|\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\| \tag{2}$$

Substituting $\boldsymbol{r}_i$ with values from Eq. (1), we obtain the volume as a trigonometric polynomial of ν:

$$\begin{aligned} 6\mathbb{V} = {}& P\sin^3\nu + Q\cos^3\nu + R\sin^2\nu\cos\nu + T\sin\nu\cos^2\nu \\ &+ U\sin^2\nu + V\cos^2\nu + W\sin\nu\cos\nu \end{aligned} \tag{3}$$

The coefficients in the polynomial depend on initial conditions, i.e., on A_i, B_i, C_i, D_i, and E_i. The exact expressions for the coefficients will be derived later. For every possible value of ν, the volume [Eq. (3)] must remain the same. In addition, it should not be equal to zero.

Now, we obtain the conditions when the trigonometric polynomial is identically constant with respect to time; i.e., the tetrahedron volume does not change. To acquire the necessary conditions for volume preservation, we will substitute different values of ν in this polynomial. Substituting $\nu = 0$, $\nu = (\pi/2)$, $\nu = \pi$, and $\nu = (3\pi/2)$ in Eq. (3) and equating all the results, we acquire the following necessary conditions on the coefficients:

$$Q + V = P + U = -Q + V = -P + U$$

so that $Q = P = 0$ and $U = V$. This leads to the simplified expression for volume:

$$6\mathbb{V} = \sin\nu\cos\nu(R\sin\nu + T\cos\nu + W) + U$$

Now, substitute $\nu = (\pi/4)$, $\nu = (3\pi/4)$, $\nu = (5\pi/4)$, and $\nu = (7\pi/4)$ to obtain

$$\begin{aligned} R + T + \sqrt{2}W &= -(R - T + \sqrt{2}W) = -R - T + \sqrt{2}W \\ &= -(-R + T + \sqrt{2}W) \end{aligned}$$

so $R + T = 0$, $R = T$ or simply $R = T = 0$, $W = 0$.

Hence, for the volume $\mathbb{V}$ of the tetrahedron to remain constant, it is necessary that

$$\begin{aligned} &P = Q = T = W = R = 0, \\ &U = V \end{aligned} \tag{4}$$

Under these conditions, the volume is equal to $\mathbb{V} = (U/6)$; hence, they are also sufficient.

Second, we want the tetrahedron to be nondegenerate. To simplify notation, we combine constants A_i, B_i, C_i, D_i, and E_i in Eq. (1) in vectors. Let $\boldsymbol{A} = \langle A_1, A_2, A_3\rangle$, $\boldsymbol{B} = \langle B_1, B_2, B_3\rangle$, $\boldsymbol{C} = \langle C_1, C_2, C_3\rangle$, $\boldsymbol{D} = \langle D_1, D_2, D_3\rangle$, and $\boldsymbol{E} = \langle E_1, E_2, E_3\rangle$. With this notation, and after appropriate simplifications, the conditions [Eq. (4)] have the form

$$\begin{aligned} &U = V \to \boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{E}\times\boldsymbol{B}), \\ &P = 0 \to \boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{D}) = 0, \\ &Q = 0 \to \boldsymbol{A}\cdot(\boldsymbol{E}\times\boldsymbol{B}) = 0, \\ &W = 0 \to \boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{E}), \\ &R = 0 \to \boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{E}) = 0, \\ &T = 0 \to \boldsymbol{A}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = 0 \end{aligned} \tag{5}$$

where $\boldsymbol{X}\cdot(\boldsymbol{Y}\times\boldsymbol{Z})$ is the mixed product of three vectors $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$.

If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$, or $\boldsymbol{E}$ is equal to zero, then $U = 0$ or $V = 0$ so that $\mathbb{V} = (U/6) = 0$, which should be avoided. If none of these vectors are equal to zero, then from Eq. (5), we can derive that all four of them should be coplanar.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are collinear, $\boldsymbol{A} = k\boldsymbol{B}$, $k \neq 0$, and

$$\begin{aligned} &U = V \to k\,\boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = \boldsymbol{C}\cdot(\boldsymbol{E}\times\boldsymbol{B}), \\ &W = 0 \to \boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = k\,\boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{E}) \end{aligned} \tag{6}$$

That implies

$$\boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = k\,\boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{E}) = -k\,\boldsymbol{C}\cdot(\boldsymbol{E}\times\boldsymbol{B}) = -k^2\boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B})$$

so that $\boldsymbol{C}\cdot(\boldsymbol{D}\times\boldsymbol{B}) = 0$ and, again, $U = 0$ and $\mathbb{V} = 0$, which is inadmissible.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are not collinear, then they form a basis in a plane; and coplanar $\boldsymbol{D}$ and $\boldsymbol{E}$ are expressed as linear combinations:

$$\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B},$$
$$\boldsymbol{E} = c\boldsymbol{A} + d\boldsymbol{B}$$

so that

$$U = V \to \boldsymbol{C}\cdot((a\boldsymbol{A} + b\boldsymbol{B})\times\boldsymbol{A}) = \boldsymbol{C}\cdot((c\boldsymbol{A} + d\boldsymbol{B})\times\boldsymbol{B}),$$
$$W = 0 \to \boldsymbol{C}\cdot((a\boldsymbol{A} + b\boldsymbol{B})\times\boldsymbol{B}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times(c\boldsymbol{A} + d\boldsymbol{B}))$$

and

$$b\boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{A}) = c\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}),$$
$$a\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = d\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

which eventually lead to $b = -c$ and $a = d$.

So, given two noncollinear vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ could be found from

$$\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B},$$
$$\boldsymbol{E} = -b\boldsymbol{A} + a\boldsymbol{B} \tag{7}$$

With such conditions, the volume $\mathbb{V}$ could be calculated from the formula

$$\mathbb{V} = \frac{b}{6}\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B})$$

The coefficient b should not be equal to zero in all subsequent calculations.

C. Shape Conservation

Before we derive conditions that ensure shape preservation, it is necessary to define what the shape is. Unfortunately, the shape of the tetrahedron does not have simple geometric or algebraic interpretation, which is partially due to the fact that the tetrahedron is not fully described by its edge lengths; i.e., the same set of edge lengths can represent several nonsimilar tetrahedra [28,29]. We can demand that the tetrahedron remains the same (i.e., it just rotates over time), but it might be too strict. We will use the following function:

$$\mathbb{Q} = 12\frac{(3\mathbb{V})^{2/3}}{\mathbb{L}}$$

that depicts how close the tetrahedron is to a regular one. Here, $\mathbb{V}$ is the volume, and $\mathbb{L}$ is the sum of squares of the tetrahedron edge lengths. For the regular tetrahedron, $\mathbb{Q} = 1$; and for the degenerate one (when four satellites lie in the same plane), $\mathbb{Q} = 0$. Figure 1 depicts several possible tetrahedra with corresponding values of $\mathbb{Q}$. This parameter $\mathbb{Q}$, which we call quality, is similar to the one used in the MMS mission [16], although it is a little different: it is not integral, and it does not take into account the size of the tetrahedron. In addition, the chosen quality is equal to the ratio of the geometric to the arithmetic means of the eigenvalues of the linear transformation operator from the regular tetrahedron to the given one [30,31].

Similar to the volume derivation [Eq. (3)] we derive the expression for $\mathbb{L}$:

$$\mathbb{L} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 + (\boldsymbol{r}_1 - \boldsymbol{r}_3)^2 + (\boldsymbol{r}_1 - \boldsymbol{r}_4)^2 + (\boldsymbol{r}_2 - \boldsymbol{r}_3)^2 + (\boldsymbol{r}_2 - \boldsymbol{r}_4)^2 + (\boldsymbol{r}_3 - \boldsymbol{r}_4)^2 = (\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 + (\boldsymbol{r}_1 - \boldsymbol{r}_3)^2 + (\boldsymbol{r}_2 - \boldsymbol{r}_3)^2 + \boldsymbol{r}_1^2 + \boldsymbol{r}_2^2 + \boldsymbol{r}_3^2 \tag{8}$$

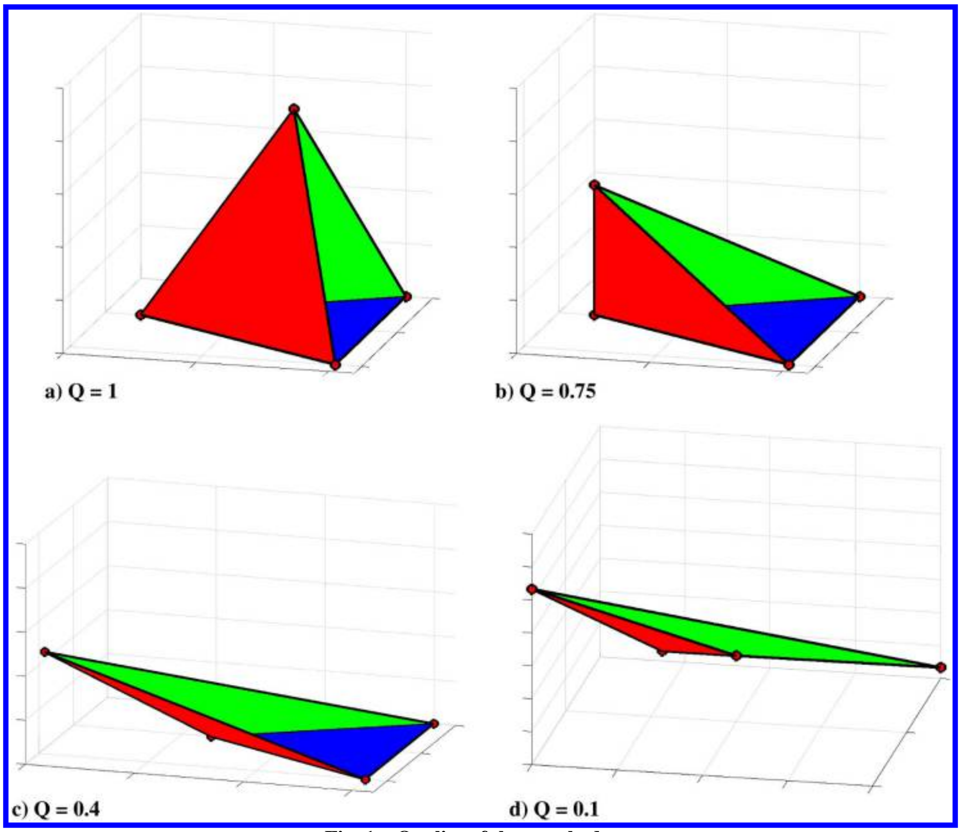


Fig. 1 Quality of the tetrahedron.

After substitutions, reductions of terms, and all simplifications, the derivation for $\mathbb{L}$ is a trigonometric polynomial. To examine the conditions for $\mathbb{L}$ to be constant, we again (at first) derive the equations for coefficients of the polynomial.

In general, the polynomial has the form

$$\mathbb{L} = P\cos^2\nu + Q\cos\nu\sin\nu + R\sin^2\nu + T\cos\nu + U\sin\nu + W \quad (9)$$

Here and after, we use the same letters of P, Q, ... to denote arbitrary coefficients not referring to the coefficients in the polynomial for volume [Eq. (3)].

The condition that $\mathbb{Q}$ is constant during the motion together with the condition of volume conservation leads to conservation of $\mathbb{L}$.

Substituting again $\nu = 0$, $\nu = (\pi/2)$, $\nu = \pi$, and $\nu = (3\pi/2)$ in Eq. (9) and equating all the results, we acquire conditions on coefficients

$$P + T + W = R + U + W = P - T + W = R - U + W$$

so that $T = U = 0$ and $P = R$. Then,

$$\mathbb{L} = Q\cos\nu\sin\nu + P + W$$

That means $Q = 0$, and so the necessary (and obviously sufficient) conditions for the conservation of $\mathbb{L}$ have the form

$$Q = T = U = 0,$$
$$P = R$$

Thus, $\mathbb{L} = P + W$.

Now, we write down these equations in more detail. Using volume conservation expressions [Eq. (7)], we can simplify the equations:

$T = U = 0$ is equal to

$$\begin{aligned}&C_1(12A_1 - 4A_2 - 4A_3) + C_2(12A_2 - 4A_1 - 4A_3)\\&\quad + C_3(12A_3 - 4A_1 - 4A_2) = 0,\\&-C_1(12B_1 - 4B_2 - 4B_3) - C_2(12B_2 - 4B_1 - 4B_3)\\&\quad - C_3(12B_3 - 4B_1 - 4B_2) = 0\end{aligned}$$

$Q = 0$ is equal to

$$\begin{aligned}&(2a^2 - 2b^2 - 6)(3A_1B_1 + 3A_2B_2 + 3A_3B_3\\&\quad - A_1B_2 - A_1B_3 - A_2B_1 - A_2B_3 - A_3B_1 - A_3B_2)\\&\quad + 2ab(3B_1^2 + 3B_2^2 + 3B_3^2 - 3A_1^2 - 3A_2^2 - 3A_3^2 + 2A_1A_2\\&\quad + 2A_1A_3 + 2A_2A_3 - 2B_1B_2 - 2B_1B_3 - 2B_2B_3) = 0\end{aligned}$$

$P = R$ is equal to

$$\begin{aligned}&a^2(3B_1^2 + 3B_2^2 + 3B_3^2 - 2B_1B_2 - 2B_1B_3 - 2B_2B_3)\\&\quad + 2ab(-3A_1B_1 - 3A_2B_2 - 3A_3B_3 + A_1B_2 + A_1B_3\\&\quad + A_2B_1 + A_2B_3 + A_3B_1 + A_3B_2)\\&\quad + b^2(3A_1^2 + 3A_2^2 + 3A_3^2 - 2A_1A_2 - 2A_1A_3 - 2A_2A_3)\\&\quad + (12A_1^2 + 12A_2^2 + 12A_3^2 - 8A_1A_2 - 8A_1A_3 - 8A_2A_3\\&\quad + 3B_1^2 + 3B_2^2 + 3B_3^2 - 2B_1B_2 - 2B_1B_3 - 2B_2B_3)\\&\quad = a^2(3A_1^2 + 3A_2^2 + 3A_3^2 - 2A_1A_2 - 2A_1A_3 - 2A_2A_3)\\&\quad + 2ab(3A_1B_1 + 3A_2B_2 + 3A_3B_3 - A_1B_2 - A_1B_3\\&\quad - A_2B_1 - A_2B_3 - A_3B_1 - A_3B_2)\\&\quad + b^2(3B_1^2 + 3B_2^2 + 3B_3^2 - 2B_1B_2 - 2B_1B_3 - 2B_2B_3)\\&\quad + (3A_1^2 + 3A_2^2 + 3A_3^2 - 2A_1A_2 - 2A_1A_3 - 2A_2A_3\\&\quad + 12B_1^2 + 12B_2^2 + 12B_3^2 - 8B_1B_2 - 8B_1B_3 - 8B_2B_3)\end{aligned}$$

All the coefficients of A_i, B_i, and C_i are from Eq. (1), and so they depend on initial values of the satellite motion.

Let us denote

$$\begin{aligned}\eta &= 3A_1B_1 + 3A_2B_2 + 3A_3B_3 - A_1B_2 - A_1B_3 - A_2B_1\\&\quad - A_2B_3 - A_3B_1 - A_3B_2,\\\zeta &= 3B_1^2 + 3B_2^2 + 3B_3^2 - 3A_1^2 - 3A_2^2 - 3A_3^2 + 2A_1A_2\\&\quad + 2A_1A_3 + 2A_2A_3 - 2B_1B_2 - 2B_1B_3 - 2B_2B_3\end{aligned}$$

so that

$$Q = 0,$$
$$P = R$$

becomes

$$(a^2 - b^2 - 3)\eta + ab\zeta = 0,$$
$$(a^2 - b^2 - 3)\zeta - 4ab\eta = 0$$

This is a system of linear equations with respect to $a^2 - b^2 - 3$ and ab. It has a trivial solution, but then $ab = 0$ and either $a = 0$ or $-b^2 - 3 = 0$, which is impossible, or $b = 0$, which makes the tetrahedron degenerate. So, for the nondegenerate constant shaped tetrahedron, the system must have a nontrivial solution; therefore, the determinant of the corresponding matrix should be equal to zero:

$$\det\begin{pmatrix}\eta & \zeta\\ \zeta & -4\eta\end{pmatrix} = -4\eta^2 - \zeta^2 = 0$$

That means $\eta = 0$ and $\zeta = 0$.

Finally, for the nondegenerate tetrahedron, preserving its volume and quality (size and shape), vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ must be noncollinear and the following expressions must be true:

$$\begin{aligned}&\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B},\\&\boldsymbol{E} = -b\boldsymbol{A} + a\boldsymbol{B},\\&3A_1B_1 + 3A_2B_2 + 3A_3B_3 - A_1B_2 - A_1B_3 - A_2B_1\\&\quad - A_2B_3 - A_3B_1 - A_3B_2 = 0,\\&3(B_1^2 + B_2^2 + B_3^2 - A_1^2 - A_2^2 - A_3^2) + 2(A_1A_2 + A_1A_3 + A_2A_3\\&\quad - B_1B_2 - B_1B_3 - B_2B_3) = 0,\\&C_1(3A_1 - A_2 - A_3) + C_2(3A_2 - A_1 - A_3)\\&\quad + C_3(3A_3 - A_1 - A_2) = 0,\\&C_1(3B_1 - B_2 - B_3) + C_2(3B_2 - B_1 - B_3)\\&\quad + C_3(3B_3 - B_1 - B_2) = 0\end{aligned} \quad (10)$$

The system splits up into three systems:

The one that contains only $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\begin{aligned}&3A_1B_1 + 3A_2B_2 + 3A_3B_3 - A_1B_2 - A_1B_3 - A_2B_1 - A_2B_3\\&\quad - A_3B_1 - A_3B_2 = 0,\\&3(B_1^2 + B_2^2 + B_3^2 - A_1^2 - A_2^2 - A_3^2) + 2(A_1A_2 + A_1A_3\\&\quad + A_2A_3 - B_1B_2 - B_1B_3 - B_2B_3) = 0\end{aligned} \quad (11)$$

The one that contains $\boldsymbol{C}$:

$$\begin{aligned}&C_1(3A_1 - A_2 - A_3) + C_2(3A_2 - A_1 - A_3)\\&\quad + C_3(3A_3 - A_1 - A_2) = 0,\\&C_1(3B_1 - B_2 - B_3) + C_2(3B_2 - B_1 - B_3)\\&\quad + C_3(3B_3 - B_1 - B_2) = 0\end{aligned} \quad (12)$$

The one that contains $\boldsymbol{D}$ and $\boldsymbol{E}$:

$$\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B},$$
$$\boldsymbol{E} = -b\boldsymbol{A} + a\boldsymbol{B} \tag{13}$$

To fully describe all possible configurations preserving volume and quality, one should solve Eq. (10) for unknown vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$. This is the system of 10 equations with 15 variables and 2 parameters of a and b. Given solutions $\boldsymbol{A}$ and $\boldsymbol{B}$, and constants a and b, one can easily calculate $\boldsymbol{D}$ and $\boldsymbol{E}$. Moreover, vector $\boldsymbol{C}$ has three components, but only two equations depend on them; so, $\boldsymbol{C}$ could be found only up to a factor. This arbitrary factor is the parameter in the general solution. The remaining system [Eq. (11)] contains six variables and two equations, and so the solutions are four-parametric families. Together with a, b, and the norm of vector $\boldsymbol{C}$, they form seven-parametric families of solutions of general system (10). Note that the satellite renumbering does not affect the dynamics, and so we refer to two different solutions obtained from each other by renumbering the satellites to a single family of solutions.

III. Particular Solutions and Reference Orbits

In a search for particular solutions, we make the following variable changes:

$$A_1 = \alpha\cos\varphi, \qquad B_1 = \alpha\sin\varphi,$$
$$A_2 = \beta\cos\psi, \qquad B_2 = \beta\sin\psi,$$
$$A_3 = \gamma\cos\theta, \qquad B_3 = \gamma\sin\theta$$

Here, α, β, and γ are amplitudes of oscillations of the first, second, and third satellites in the ORF, respectively; and φ, ψ, and θ are the initial phases.

System (11) transforms into

$$2(\alpha\beta\cos(\varphi+\psi) + \alpha\gamma\cos(\varphi+\theta) + \beta\gamma\cos(\psi+\theta))$$
$$= 3(\alpha^2\cos 2\varphi + \beta^2\cos 2\psi + \gamma^2\cos 2\theta),$$
$$2(\alpha\beta\sin(\varphi+\psi) + \alpha\gamma\sin(\varphi+\theta) + \beta\gamma\sin(\psi+\theta))$$
$$= 3(\alpha^2\sin 2\varphi + \beta^2\sin 2\psi + \gamma^2\sin 2\theta) \tag{14}$$

As we said, the solutions to Eq. (11) are four-parametric families. First of all, we note that variable changes of $\alpha \to N\cdot\alpha$, $\beta \to N\cdot\beta$, $\gamma \to N\cdot\gamma$, and $\varphi \to \varphi+\xi$, $\theta \to \theta+\xi$, $\psi \to \psi+\xi$ do not change system (11), and so two of the four parameters are arbitrary factors for the amplitudes and the arbitrary angle, which is the initial phase of the motion. To simplify the system, we choose an initial moment of time so that $\varphi \to 0$, i.e., choose $\xi = -\varphi$.

Given values of α/β, β/γ, and γ one could solve Eq. (11) at least numerically. So, three amplitudes of α, β, and γ actually define the motion. Thus, the solutions of (10) are seven-parametric families, where parameters are: a, b, the norm of $\boldsymbol{C}$, the phase ξ and three independent relations between α, β, γ, for example, the norm of (α, β, γ), α/β and β/γ.

Unfortunately, we were not able to obtain a general solution of the system, and so we consider some important special cases: two of the three amplitudes are equal. That is, we set $\beta/\gamma = 1$ so the $\beta = \gamma = K$. If $K = 0$, three satellites (the second, the third, and the fourth) are resting in the ORF (i.e., all of them are located on the y axis), and so the tetrahedron is degenerate; thus, $K \neq 0$. Finally, under all assumptions and after all substitutions,

$$\beta = \gamma = K \neq 0,$$
$$\alpha = pK,$$
$$\varphi = 0$$

System (12) transforms into

$$C_1(3p - \cos\psi - \cos\theta) + C_2(3\cos\psi - p - \cos\theta)$$
$$+ C_3(3\cos\theta - p - \cos\psi) = 0,$$
$$C_1(-\sin\psi - \sin\theta) + C_2(3\sin\psi - \sin\theta)$$
$$+ C_3(3\sin\theta - \sin\psi) = 0$$

and system (11) transforms into

$$2p(\cos\psi + \cos\theta) + 2\cos(\psi+\theta) = 3p^2 + 3(\cos 2\psi + \cos 2\theta),$$
$$2p(\sin\psi + \sin\theta) + 2\sin(\psi+\theta) = 3(\sin 2\psi + \sin 2\theta)$$

After simplifications, we have the following:

$$4\cos\frac{\psi+\theta}{2}\left(p\cos\frac{\psi-\theta}{2} + 4\cos\frac{\psi+\theta}{2} - 6\cos\frac{\psi+\theta}{2}\cos^2\frac{\psi-\theta}{2}\right)$$
$$+ 12\cos^2\frac{\psi-\theta}{2} - 8 - 3p^2 = 0,$$
$$4\sin\frac{\psi+\theta}{2}\left(p\cos\frac{\psi-\theta}{2} + 4\cos\frac{\psi+\theta}{2} - 6\cos\frac{\psi+\theta}{2}\cos^2\frac{\psi-\theta}{2}\right) = 0 \tag{15}$$

The second equation in the system allows us to distinguish the following two cases:

A. Case 1: $\sin((\psi+\theta)/2) = 0$

In this case, $\psi = -\theta + 2\pi k$. All angles are modulo 2π, and so $\psi = -\theta$. System (15) has the form

$$4p\cos\theta - 12\cos^2\theta + 8 - 3p^2 = 0,$$
$$\psi = -\theta$$

Hence,

$$\cos\theta = \frac{p \pm 2\sqrt{6-2p^2}}{6}$$

This equation has solutions when $p \in [-\sqrt{3}, \sqrt{3}]$ but, due to nonnegativity of p, we have two families of solutions (signs plus and minus):

$$\psi = -\arccos\frac{p \pm 2\sqrt{6-2p^2}}{6},$$
$$\theta = \arccos\frac{p \pm 2\sqrt{6-2p^2}}{6} \tag{16}$$

This also means that $\sin\psi = -\sin\theta$ and $\cos\psi = \cos\theta$. If $p = \sqrt{3}$, two families coincide.

Using these results, we could transform system (12) into

$$p(3C_1 - C_2 - C_3) + (3C_2 - C_1 - C_3)\cos\psi$$
$$+ (3C_3 - C_1 - C_2)\cos\theta = 0,$$
$$(3C_2 - C_1 - C_3)\sin\psi + (3C_3 - C_1 - C_2)\sin\theta = 0$$

where $\sin\psi = -\sin\theta$ implies $C_2 = C_3$, and

$$C_1 = C_2\frac{2p - 4\cos\theta}{3p - 2\cos\theta}$$

Finally, we have the solution

$$\boldsymbol{A} = K\begin{pmatrix} p \\ \cos\theta \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{B} = K\begin{pmatrix} 0 \\ -\sin\theta \\ \sin\theta \end{pmatrix}, \quad \boldsymbol{C} = c\begin{pmatrix} \frac{2p-4\cos\psi}{3p-2\cos\psi} \\ 1 \\ 1 \end{pmatrix}$$

where $K > 0$, and c are arbitrary coefficients.

B. Case 2: $\sin((\psi+\theta)/2) \neq 0$

In this case, system (15) has the form

$$\cos^2\frac{\psi-\theta}{2} = \frac{8+3p^2}{12},$$
$$p\cos\frac{\psi-\theta}{2} = \frac{3p^2}{2}\cos\frac{\psi+\theta}{2}$$

If $p \neq 0$, then this system has solutions when

$$p \in \left[\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$$

They are

$$\psi = \pm\arccos\frac{(4/p)+(3p/2)-\sqrt{6-2(2/p-3p)^2}}{9},$$
$$p \in \left[\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right],$$
$$\theta = \pm\arccos\frac{(4/p)+(3/2)+\sqrt{6-2(2/p-3p)^2}}{9},$$
$$p \in \left[\frac{2}{3}, \frac{2}{\sqrt{3}}\right],$$
$$\theta = \mp\arccos\frac{(4/p)+(3p/2)+\sqrt{6-2(2/p-3p)^2}}{9},$$
$$p \in \left[\frac{1}{\sqrt{3}}, \frac{2}{3}\right]$$

Cases 1 and 2 give different solution families if only

$$\frac{p\pm 2\sqrt{6-2p^2}}{6} \neq \frac{(4/p)+(3p/2)\pm\sqrt{6-2(2/p-3p)^2}}{9}$$

which is equivalent to $p \neq 1$ and $p = (1/\sqrt{3})$.

Explicit expressions for C_i are quite bulky and not presented. The case of $p = 0$ will be considered in the following.

IV. Solutions Analysis

A. One Zero Amplitude

In this subsection, we assume that $p = 0$ so that $\beta = \gamma = K \neq 0$, $\alpha = 0$. Two satellites have the same amplitude, and the third one has zero amplitude; i.e., it rests in the ORF. In the IRF, two satellites of the tetrahedron move along the same orbit with constant shift.

In case 1, we have

$$\psi = -\arccos\frac{\pm\sqrt{6}}{3}, \qquad \theta = \arccos\frac{\pm\sqrt{6}}{3}$$

But, zero amplitude implies that two satellites rest in the ORF, and so the phase shift between two other satellites depends only on $|\theta-\psi|$. But,

$$|\theta-\psi| = \left|2\arccos\frac{\pm\sqrt{6}}{3}\right| = \arccos\frac{1}{3}$$

Thus, two solutions coincide. After that, we obtain $C_2 = C_3$, and $C_1 = 2C_2$.

We note that phase φ was chosen to be equal to zero just for simplicity; for a complete description of the solution, we should return this arbitrary coefficient. The full set of initial values is given by

$$\boldsymbol{A} = K\begin{pmatrix} 0 \\ \sqrt{6}/3\cos\varphi + \sqrt{3}/3\sin\varphi \\ \sqrt{6}/3\cos\varphi - \sqrt{3}/3\sin\varphi \end{pmatrix},$$
$$\boldsymbol{B} = K\begin{pmatrix} 0 \\ -\sqrt{3}/3\cos\varphi + \sqrt{6}/3\sin\varphi \\ \sqrt{3}/3\cos\varphi + \sqrt{6}/3\sin\varphi \end{pmatrix},$$
$$\boldsymbol{C} = c\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B}, \qquad \boldsymbol{E} = -b\boldsymbol{A} + a\boldsymbol{B} \tag{17}$$

where a and φ are arbitrary coefficients; and b, c, and K are arbitrary nonzero coefficients. In this case,

$$\mathbb{V} = \frac{b}{6}\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B}) = cK^2 b\frac{2\sqrt{2}}{9}, \quad \mathbb{L} = \frac{8}{3}K^2(a^2+b^2+5)+8c^2$$

The maximum of quality is achieved when $a = 0$, $b = \pm\sqrt{5}$, and $c = \pm K\sqrt{(5/3)}$; and it is equal to $\mathbb{Q}_{\max} = (1/\sqrt[3]{5})$

Case 2 does not have solutions because

$$p \notin \left[\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$$

but setting $p = 0$ in the general case implies

$$\cos^2\frac{\psi-\theta}{2} = \frac{2}{3},$$
$$\sin\frac{\psi+\theta}{2} \neq 0$$

This means that $\cos(\psi-\theta) = (1/3)$, and this solution is the same as in Case 1.

Figure 2 shows the visualization of the resulting tetrahedron: as we can see, it just rotates around one of its edges.

The next figures show the evolution of the formation quality in the nonlinear motion model including the J_2 geopotential harmonic, depending on the initial conditions and the size of the satellite relative orbits. It is necessary to mention that, despite the fact that in the linear model of motion the quality factor does not depend on φ, in real motion, the difference in initial conditions yields different relative drift and a different degrading rate of the tetrahedron.

The different subfigures of Fig. 3 depict different initial values (different φ parameter with step $\pi/2$) for the formation of the same size ($K = 100$ m) in the same orbit. Figure 4 contains the same information in one combined figure for a more convenient comparison. The same information for $K = 1000$ m is presented in Figs. 5 and 6, respectively.

The different subfigures of Fig. 7 contain the information of the tetrahedron degrading for different tetrahedron sizes, and the initial condition φ is chosen so that the degrading rate is minimal; the black line represents the conserving quality in the linear model. Figure 8 contains the same information combined into one figure. As we can

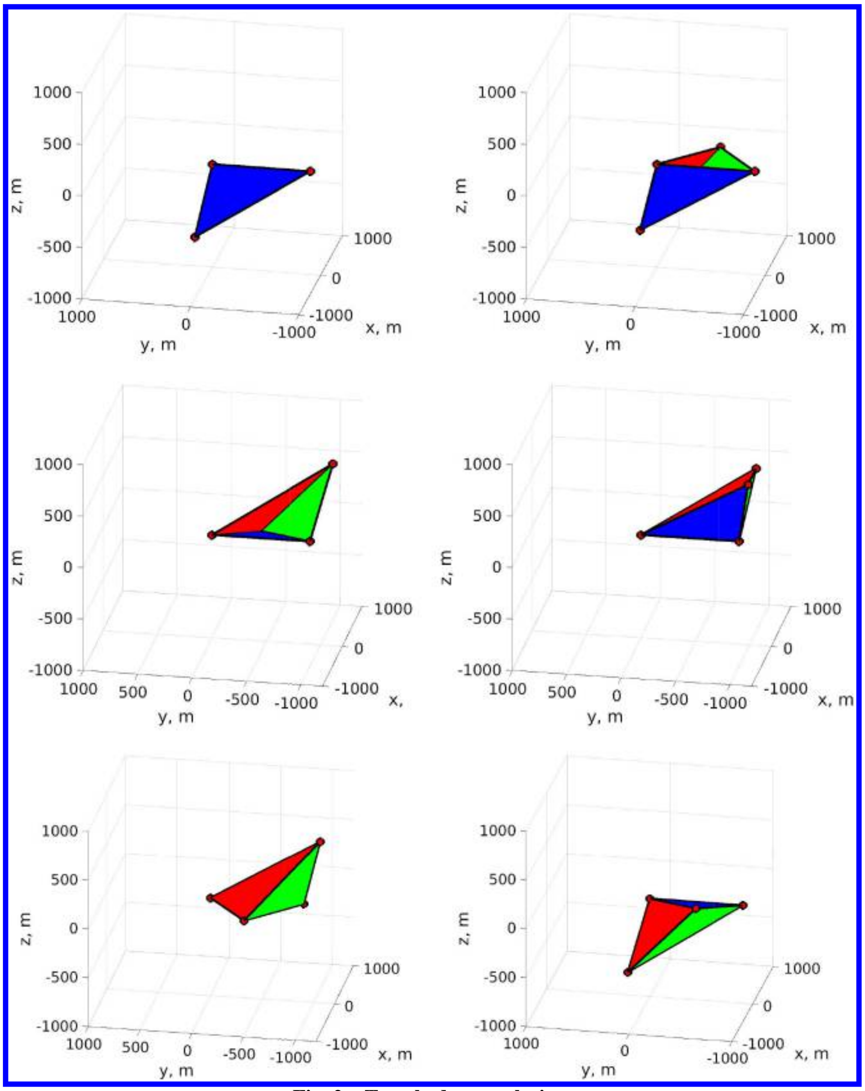


Fig. 2 Tetrahedron evolution.

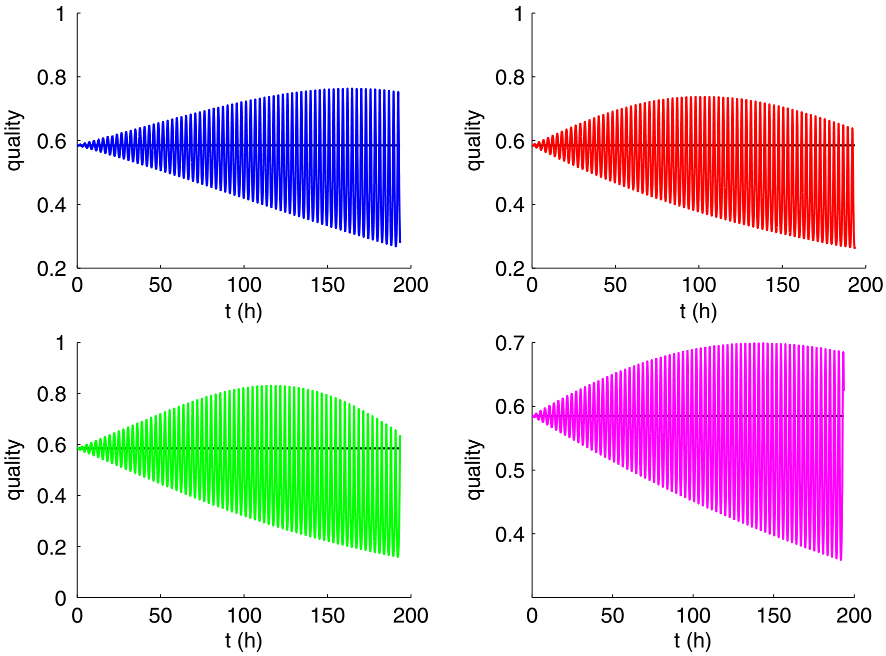


Fig. 3 Orbit radius of 10,000 km.

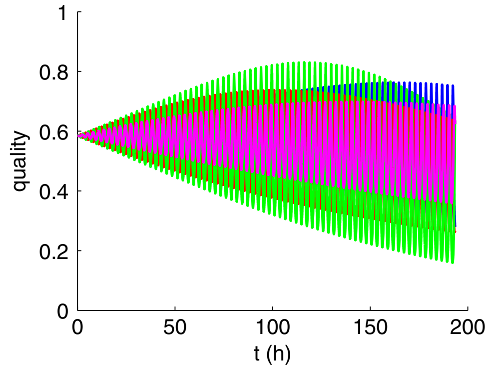


Fig. 4 Orbit radius of 10,000 km, $K = 100$, and different values of φ.

see, the degradation rate increases with formation size, which is expected due to the increased value of disturbances.

The difference in chief inclination could also greatly affect the degrading rate of the corresponding tetrahedron. The different subfigures of Fig. 9 contain the information of the tetrahedron degrading for different initial inclinations. Figure 10 contains the same information combined into one figure.

We also note that such a configuration with two satellites moving on the same orbit (leader–follower formation) is the unique solution to system (10) without any additional assumptions on the remaining motion parameters. To show that, we assume that $\alpha = 0$; using rotational symmetry, we look for initial conditions when $\psi = 0$. Substituting into Eq. (11), we have

$$3\gamma^2 \cos\theta \sin\theta - \beta\gamma \sin\theta = 0,$$
$$2\beta\gamma \cos\theta = 3\beta^2 + 3\gamma^2 \cos 2\theta$$

The first equation gives either $\gamma = 0$, $\sin\theta = 0$, or $3\gamma \cos\theta = \beta$. The first option implies that three satellites have zero amplitude (the one in the origin of the ORF, the assumed one and the one with amplitude $\gamma = 0$) and therefore the tetrahedron is degenerate. The second option implies that four satellites lie in a plane and tetrahedron is again degenerate.

For a nondegenerate formation, $\cos\theta = (\beta/3\gamma)$; substituting this into the second equation yields

$$2\beta\gamma \frac{\beta}{3\gamma} = 3\beta^2 + 3\gamma^2 \left(2\frac{\beta^2}{9\gamma^2} - 1\right)$$

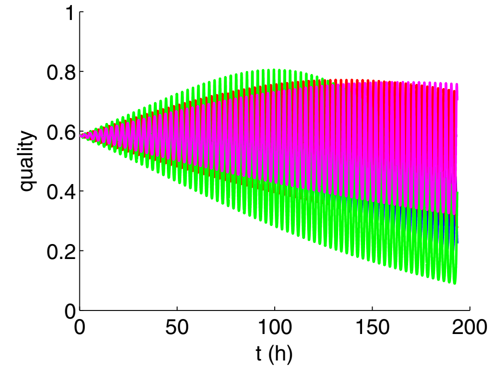


Fig. 6 Orbit radius of 10,000 km, $K = 1000$, and different values of φ.

or simply $\beta = \gamma$. So, the leader–follower formation for two of the four satellites implies equality of the two amplitudes of the remaining two satellites.

B. Three Equal Amplitudes

In this subsection, we assume that $p = 1$ and $\alpha = \beta = \gamma = K \neq 0$. Cases 1 and 2 give the same solutions; from Eq. (16), we obtain

$$\psi = -\arccos\frac{1 \pm 4}{6},$$
$$\theta = \arccos\frac{1 \pm 4}{6}$$

Two different families of solutions are generated by

$$(\varphi, \psi, \theta) = \left(0, \frac{4\pi}{3}, \frac{2\pi}{3}\right),$$
$$(\varphi, \psi, \theta) = \left(0, -\arccos\frac{5}{6}, \arccos\frac{5}{6}\right)$$

Every solution family is obtained by rearranging the satellites among themselves and shifting all the phases by an arbitrary number.

1. $(\varphi, \psi, \theta) = (\varphi, 4\pi/3 + \varphi, 2\pi/3 + \varphi)$

In this case, the phases are uniformly distributed on the circle; the full set of initial values is given by

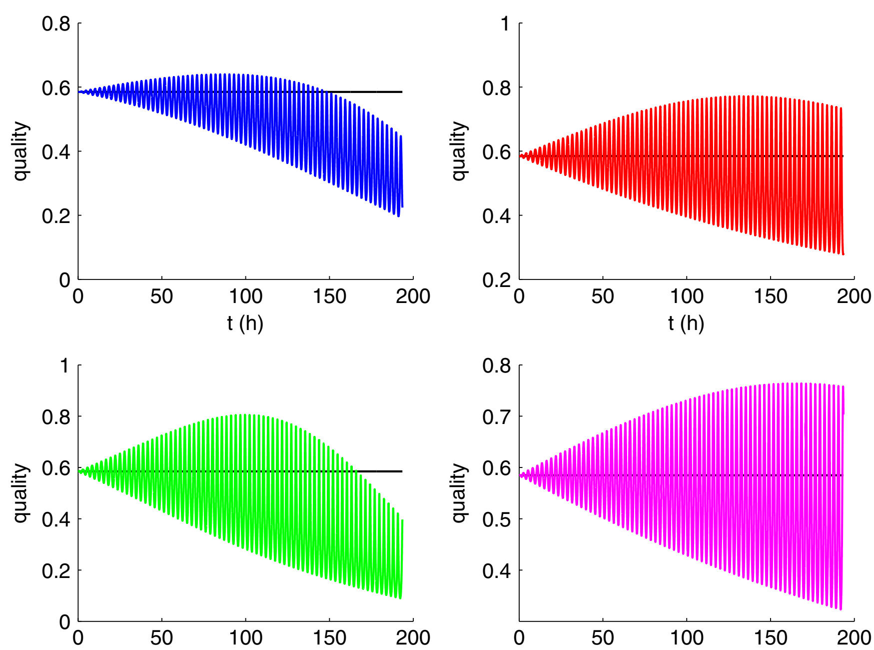
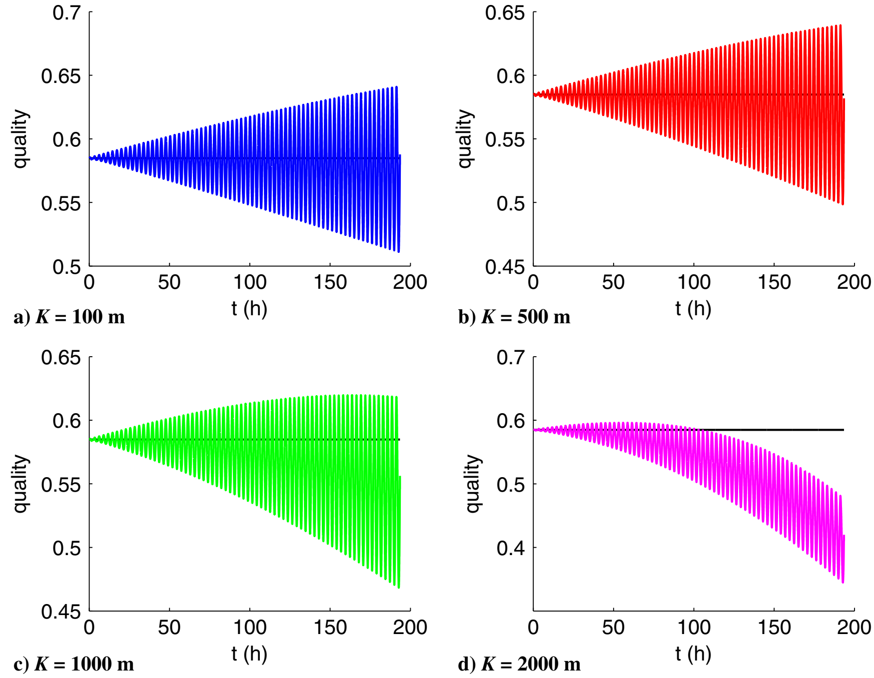


Fig. 5 Orbit radius of 10,000 km.



a) $K = 100$ m

b) $K = 500$ m

c) $K = 1000$ m

d) $K = 2000$ m

Fig. 7 Orbit radius of 10,000 km and different values of *K*.

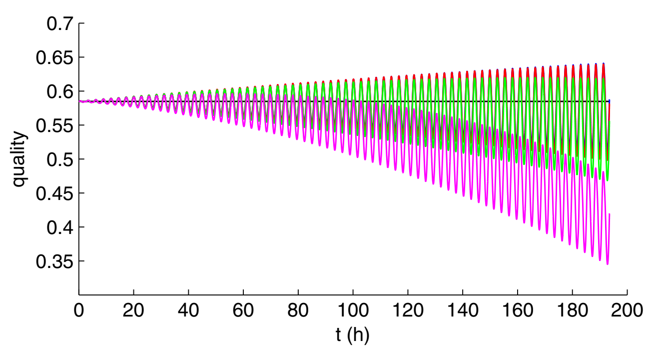


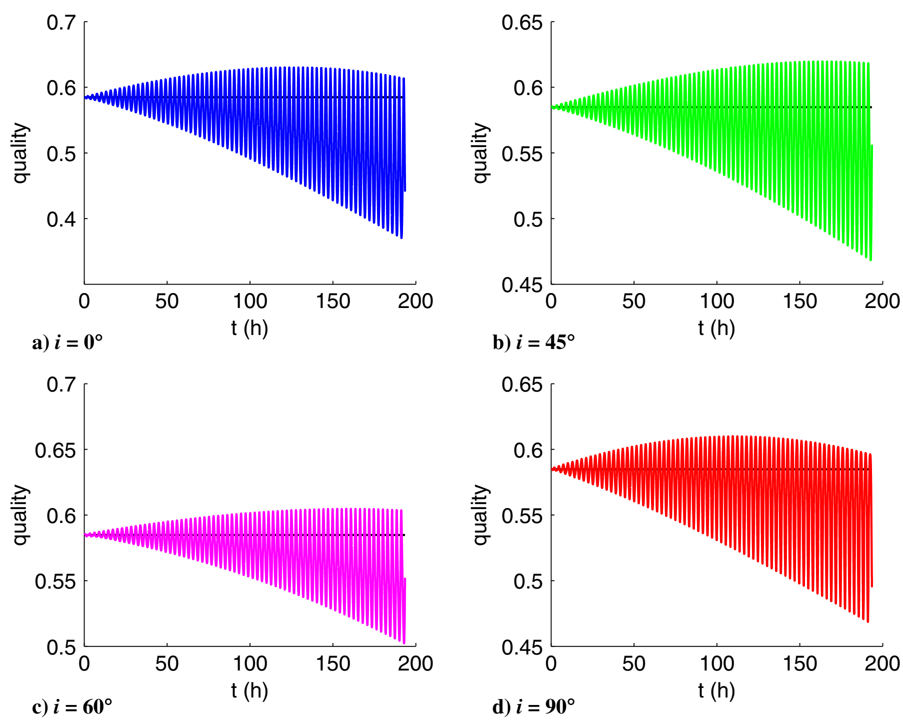
Fig. 8 Orbit radius of 10,000 km and different values of *K*.

$$
\boldsymbol{A} = K\begin{pmatrix} \cos\varphi \\ -1/2\cos\varphi + \sqrt{3}/2\sin\varphi \\ -1/2\cos\varphi - \sqrt{3}/2\sin\varphi \end{pmatrix},
$$

$$
\boldsymbol{B} = K\begin{pmatrix} \sin\varphi \\ -\sqrt{3}/2\cos\varphi - 1/2\sin\varphi \\ \sqrt{3}/2\cos\varphi - 1/2\sin\varphi \end{pmatrix},
$$

$$
\boldsymbol{C} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
$$

$$
\boldsymbol{D} = a\boldsymbol{A} + b\boldsymbol{B}, \qquad \boldsymbol{E} = -b\boldsymbol{A} + a\boldsymbol{B} \tag{18}
$$



a) $i = 0°$

b) $i = 45°$

c) $i = 60°$

d) $i = 90°$

Fig. 9 Orbit radius of 10,000 km and different inclinations.

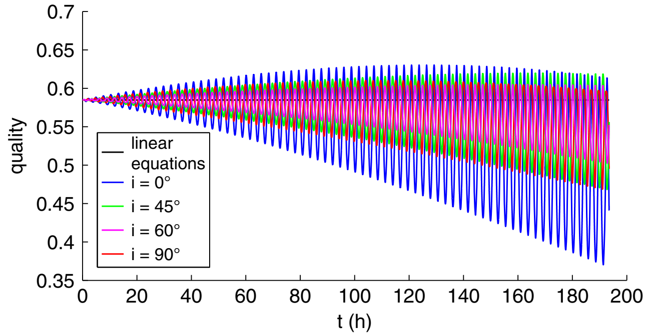


Fig. 10 Orbit radius of 10,000 km and different inclinations.

We emphasize again that this is a case when parameters $p = 1$ and $\beta/\gamma = 1$, and so the solution is a five-parametric family with φ, a, b, c, and K as parameters. The family also includes all possible rearrangements of the satellites.

For this family, the volume and edge lengths are equal to

$$\mathbb{V} = \frac{b}{6}\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B}) = -cK^2 b\frac{\sqrt{3}}{4} \quad \mathbb{L} = 6K^2(a^2+b^2+5)+3c^2$$

To maximize quality, we set $a = 0$ so that

$$\mathbb{Q} = 3\sqrt[3]{4}\cdot\frac{(c/K)^{2/3}b^{2/3}}{2b^2+10+(c/K)^2}$$

Maximum quality is obtained when $b = \pm\sqrt{5}$ and $c = \pm K\sqrt{10}$, and so $\mathbb{Q}_{\max} = (1/\sqrt[3]{5})$.

Figure 11 shows the visualization of the resulting tetrahedron; and Fig. 12 shows the evolution of the formation quality in the nonlinear motion model, depending on the relative orbits of the satellites.

2. $(\varphi,\psi,\theta) = (\varphi, -\arccos(5/6)+\varphi, \arccos(5/6)+\varphi)$

In this case, the phases are shifted on the angle $\arccos(5/6)\approx 33.56$ deg, and the full set of initial values is given by

$$\boldsymbol{A} = K\begin{pmatrix}\cos\varphi\\ 5/6\cos\varphi+\sqrt{11}/6\sin\varphi\\ 5/6\cos\varphi-\sqrt{11}/6\sin\varphi\end{pmatrix},$$

$$\boldsymbol{B} = K\begin{pmatrix}\sin\varphi\\ -\sqrt{11}/6\cos\varphi+5/6\sin\varphi\\ \sqrt{11}/6\cos\varphi+5/6\sin\varphi\end{pmatrix},$$

$$\boldsymbol{C} = c\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix},$$

$$\boldsymbol{D} = a\boldsymbol{A}+b\boldsymbol{B}, \qquad \boldsymbol{E} = -b\boldsymbol{A}+a\boldsymbol{B} \tag{19}$$

where φ, a, b, c, and K are arbitrary coefficients. In this case,

$$\mathbb{V} = \frac{b}{6}\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B}) = -cK^2 b\frac{11\sqrt{11}}{108},$$
$$\mathbb{L} = \frac{22}{9}K^2(a^2+b^2+5)+11c^2$$

The set $a = 0$, $b = \pm\sqrt{5}$, and $c = \pm K\sqrt{10}/3$ maximizes quality: $\mathbb{Q}_{\max} = (1/\sqrt[3]{5})$.

Figure 13 shows the visualization of the resulting tetrahedron; and Fig. 14 shows the evolution of the formation quality in the nonlinear motion model, depending on the relative orbits of the satellites.

It is easy to see that, as in previous cases, the rate of formation degeneration increases with relative distances increasing.

V. Nonlinear Error Effects

The Clohessy–Wiltshire equations being used in the previous sections are obtained through linearization of the Keplerian motion; thus, there is increasing over time discrepancy between true trajectory of the satellite and linear approximation. This implies differences between the ideal linear motion with the constant tetrahedron quality and the nonlinear motion and evolution of the tetrahedron. Partially, this fact explains the dependence of quality degradation on the initial phase of the satellite. In this section, we will explore how nonlinear terms (but not the eccentricity) affect the tetrahedron.

To do that, we expand the equations of relative motion on a near-circular orbit up to the second order on the small parameter, which is the ratio between relative distances and the radius of orbit

$$\ddot{x}-2n\dot{y}-n^2x = -\frac{\mu(\rho+x)}{((\rho+x)^2+y^2+z^2)^{3/2}}+\frac{\mu}{\rho^2},$$
$$\ddot{y}+2n\dot{x}-n^2y = -\frac{\mu y}{((\rho+x)^2+y^2+z^2)^{3/2}},$$
$$\ddot{z} = -\frac{\mu z}{((\rho+x)^2+y^2+z^2)^{3/2}},$$

Dropping the terms of the third and higher orders on $\boldsymbol{r}/\rho$, we obtain the perturbed Clohessy–Wiltshire equations [32]:

$$\ddot{x}-2n\dot{y}-n^2x = 2n^2x-\frac{3n^2}{\rho}\left(x^2-\frac{y^2}{2}-\frac{z^2}{2}\right),$$
$$\ddot{y}+2n\dot{x}-n^2y = -n^2y+\frac{n^2}{\rho}(3xy),$$
$$\ddot{z} = -n^2z+\frac{n^2}{\rho}(3xz),$$

To bring these equations to a suitable form, we first transform them into dimensionless ones. Let $\bar{x} = x/\rho$, $\bar{y} = y/\rho$, $\bar{z} = z/\rho$, and $\nu = nt$; and $(\cdot)'$ is differentiating with respect to ν:

$$\bar{x}''-2\bar{y}'-3\bar{x} = -3\bar{x}^2+\frac{3}{2}\bar{y}^2+\frac{3}{2}\bar{z}^2,$$
$$\bar{y}''+2\bar{x}' = 3\bar{x}\,\bar{y},$$
$$\bar{z}''+\bar{z} = 3\bar{x}\,\bar{z}$$

and, finally, we carry out the small parameter $\varepsilon = K/\rho$ in the explicit form: $X = \bar{x}/\varepsilon$, $Y = \bar{y}/\varepsilon$, and $Z = \bar{z}/\varepsilon$:

$$X''-2Y'-3X = \varepsilon\left(-3X^2+\frac{3}{2}Y^2+\frac{3}{2}Z^2\right),$$
$$Y''+2X' = \varepsilon(3XY),$$
$$Z''+Z = \varepsilon(3XZ) \tag{20}$$

We solve these equations only up to the first order on ε; to do so, we assume that Eq. (20) has the solution in the following form:

$$\boldsymbol{R} = \langle X, Y, Z\rangle = \boldsymbol{R}_0+\varepsilon\boldsymbol{R}_1+\ldots$$

where ... represents all omitted members of the series. Because $\varepsilon \ll 1$, we can use the method of perturbations. If $\varepsilon = 0$, we obtain the Clohessy–Wiltshire equations; hence, $\boldsymbol{R}_0 = \langle X_0, Y_0, Z_0\rangle$ is just the solution [Eq. (1)] to these. For $\boldsymbol{R}_1 = \langle X_1, Y_1, Z_1\rangle$, we have

$$X_1''-2Y_1'-3X_1 = \varepsilon\left(-3X_0^2+\frac{3}{2}Y_0^2+\frac{3}{2}Z_0^2\right),$$
$$Y_1''+2X_1' = \varepsilon(3X_0Y_0),$$
$$Z_1''+Z_1 = \varepsilon(3X_0Z_0) \tag{21}$$

These are nonhomogeneous second-order linear differential equations with constant coefficients; hence, they can be easily solved. The solution $\boldsymbol{R}_0+\varepsilon\boldsymbol{R}_1$ is the next order approximation as compared to the Clohessy–Wiltshire equations.

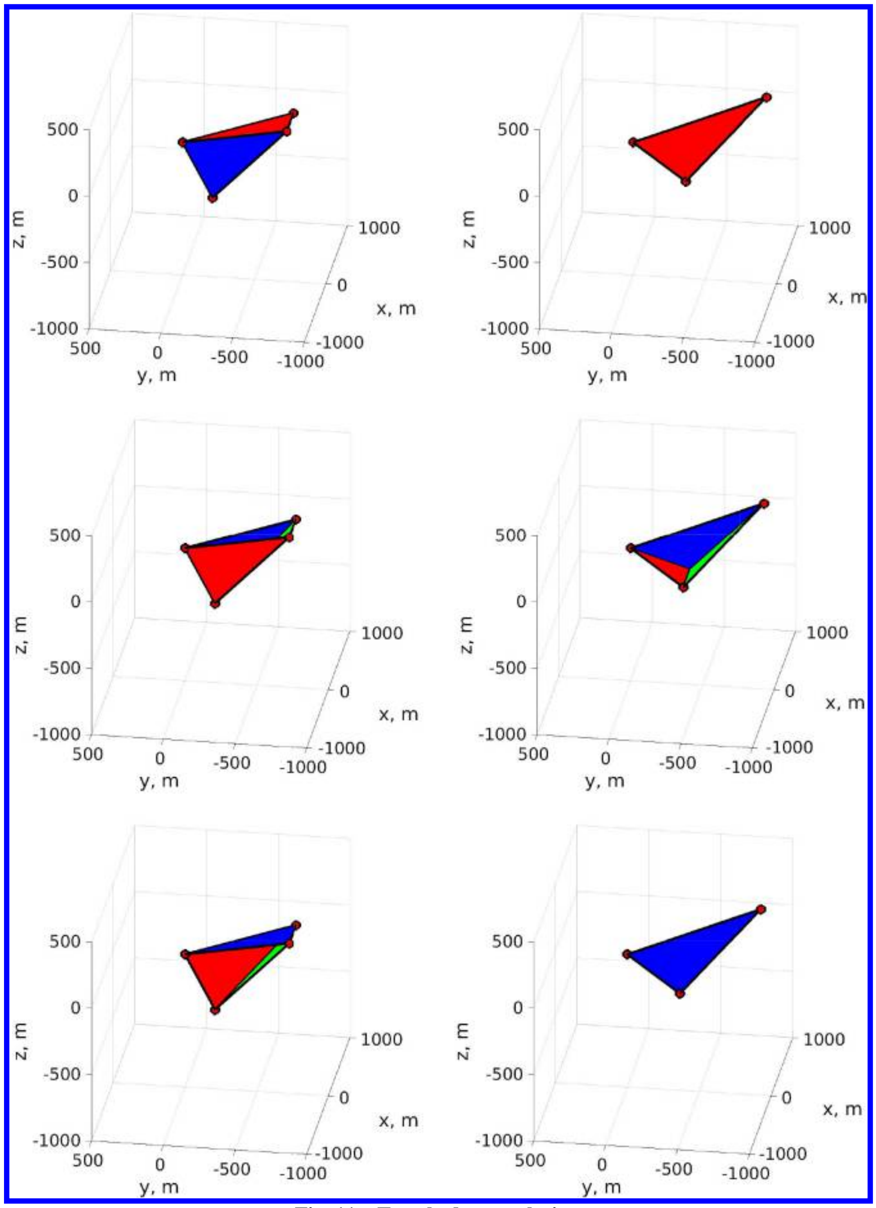


Fig. 11 Tetrahedron evolution.

The solutions in closed form can be found in a paper [33]; here, we just substitute them into expressions for the volume [Eq. (2)] and sum of the lengths squared [Eq. (8)]. If $\varepsilon = 0$, these expressions become constants; for $\varepsilon \neq 0$, they depend on ε, and this dependence is the error between true and linear trajectories.

As we are interested only in the first-order approximation with respect to ε, we omit all members of order ε^2 and higher in the series for $\mathbb{V}$ and $\mathbb{L}$.

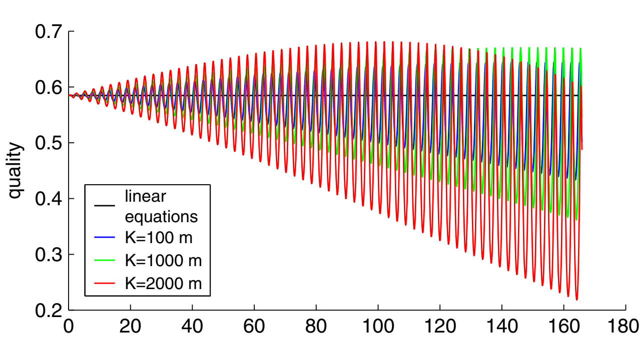


Fig. 12 Orbit radius of 10,000 km and different values of *K*.

We apply this technique for two specific families of solutions separately.

A. Family 1

For the first family of solutions [Eq. (17)] with an optimal choice of parameters a, b, c and arbitrary K, φ, we obtain the following equations:

$$
\begin{aligned}
\mathbb{V}(\nu) = {} & \frac{10\sqrt{6}}{27}K^3 + \varepsilon K^3\left(\frac{35}{24}\sin 3\varphi - \frac{15}{8}\sin\varphi\right) \\
& - \varepsilon K^3\frac{20\sqrt{10}}{9}\nu \\
& + \varepsilon K^3\left(-\frac{25}{24}\sin(3\varphi + 2\nu) + \frac{5}{4}\sin(3\varphi + 3\nu)\right. \\
& \quad + \frac{455}{108}\sin(\varphi + \nu) - \frac{25}{9}\sin(\varphi - \nu) \\
& \quad + \frac{425}{216}\sin(\varphi - 2\nu) - \frac{55}{36}\sin(\varphi + 2\nu) - \frac{5}{3}\sin(3\varphi + \nu) \\
& \quad \left. + \frac{40\sqrt{10}}{9}\sin\nu - \frac{10\sqrt{10}}{9}\sin 2\nu\right)
\end{aligned}
$$

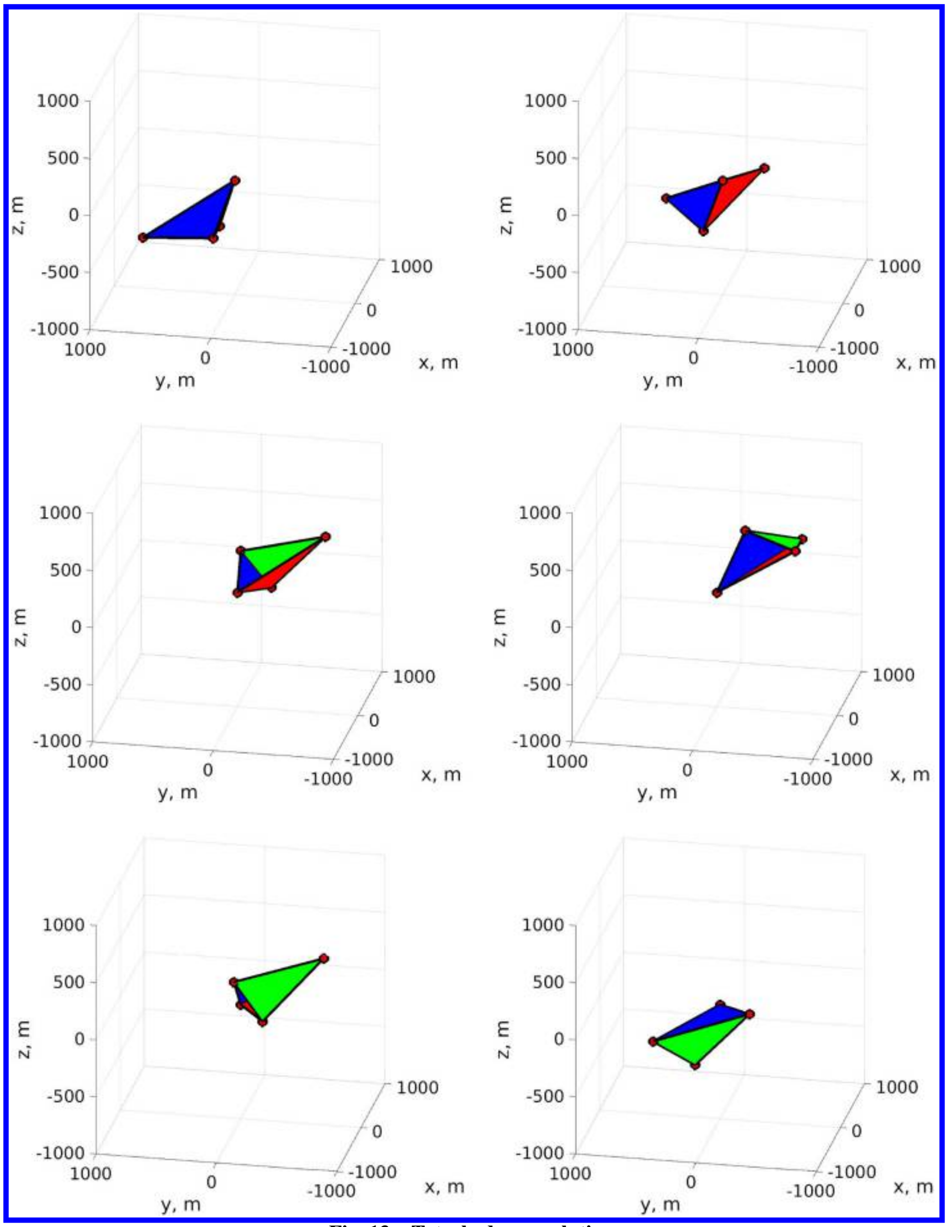


Fig. 13 Tetrahedron evolution.

$$\mathbb{L}(\nu) = 40K^2 + \varepsilon K^2\left(\frac{35\sqrt{6}}{2}\sin 3\varphi - \frac{45\sqrt{6}}{2}\sin\varphi\right) - \varepsilon K^2\frac{160\sqrt{15}}{3}\nu + \varepsilon K^2\left(\frac{160\sqrt{15}}{3}\sin\nu + 20\sqrt{6}\sin(\varphi-\nu) - \frac{335\sqrt{6}}{18}\sin(\varphi-2\nu) + 11\sqrt{6}\sin(\varphi+2\nu) - 20\sqrt{6}\sin(3\varphi+\nu) + 16\sqrt{15}\sin 2\nu - \frac{\sqrt{6}}{2}\sin(3\varphi+2\nu) + 3\sqrt{6}\sin(3\varphi+3\nu) + \frac{91\sqrt{6}}{9}\sin(\varphi+\nu)\right) + \varepsilon K^2\nu\left(12\sqrt{6}\cos(3\varphi+\nu) - \frac{88\sqrt{6}}{3}\cos(\varphi+\nu) - 32\sqrt{15}\cos\nu - 20\sqrt{6}\cos(\varphi-\nu)\right)$$

The equations, being the correct approximation to quality evolution, are still not very convenient and practical. We are interested now in obtaining linear approximation. Let

$$\bar{\mathbb{V}}(\nu) = \mathbb{V}(0) + \nu\lim_{\nu\to+\infty}\sup\frac{\mathbb{V}(\nu)-\mathbb{V}(0)}{\nu-0}$$

$$\underline{\mathbb{V}}(\nu) = \mathbb{V}(0) + \nu\lim_{\nu\to+\infty}\inf\frac{\mathbb{V}(\nu)-\mathbb{V}(0)}{\nu-0}$$

$$\hat{\mathbb{V}}(\nu) = \frac{1}{2}(\bar{\mathbb{V}}(\nu) + \underline{\mathbb{V}}(\nu))$$

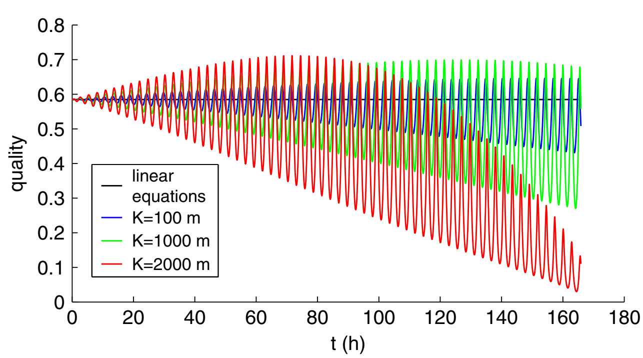


Fig. 14 Orbit radius of 10,000 km.

The difference between two linear functions depicts the amplitude of volume oscillations: upper limit approximate maximal, lower limit/minimal values, and $\hat{\mathbb{V}}$ as the average linear approximation. Similarly,

$$\bar{\mathbb{L}}(\nu) = \mathbb{L}(0) + \nu \lim_{\nu\to+\infty} \sup \frac{\mathbb{L}(\nu) - \mathbb{L}(0)}{\nu - 0}$$

$$\underline{\mathbb{L}}(\nu) = \mathbb{L}(0) + \nu \lim_{\nu\to+\infty} \inf \frac{\mathbb{L}(\nu) - \mathbb{L}(0)}{\nu - 0}$$

$$\hat{\mathbb{L}}(\nu) = \frac{1}{2}(\bar{\mathbb{L}}(\nu) + \underline{\mathbb{L}}(\nu))$$

We have

$$\hat{\mathbb{V}}(\nu) = \bar{\mathbb{V}}(\nu) = \underline{\mathbb{V}}(\nu) = \frac{10\sqrt{6}}{27}K^3 - \varepsilon K^3 \frac{20\sqrt{10}}{9}\nu$$

$$\bar{\mathbb{L}}(\nu) = 40K^2 + \nu K^2\left(-\varepsilon \frac{160\sqrt{15}}{3} + \varepsilon M_{\max}\right)$$

$$\underline{\mathbb{L}} = 40K^2 + \nu K^2\left(-\varepsilon \frac{160\sqrt{15}}{3} + \varepsilon M_{\min}\right)$$

where $M_{\max}$ and $M_{\min}$ are the maximal and minimal values of

$$\left(12\sqrt{6}\cos(3\varphi+\nu) - \frac{88\sqrt{6}}{3}\cos(\varphi+\nu) - 32\sqrt{15}\cos\nu - 20\sqrt{6}\cos(\varphi-\nu)\right)$$

This is the expression of the form $A_\varphi \cos\nu + B_\varphi \sin\nu$, where A_φ and B_φ are coefficients depending on φ. Its maximum over ν is equal to $\sqrt{A_\varphi^2 + B_\varphi^2}$, and the minimum is equal to $-\sqrt{A_\varphi^2 + B_\varphi^2}$:

$$\hat{\mathbb{L}}(\nu) = 40K^2 - K^2\varepsilon\nu \frac{160\sqrt{15}}{3}$$

So, the average rate of $\mathbb{L}$ is constant in the first approximation:

$$\hat{\mathbb{Q}}(\nu) = 12\frac{(3\hat{\mathbb{V}})^{(2/3)}}{\hat{\mathbb{L}}} = \frac{1}{\sqrt[3]{5}}$$

Figures 15–17 depict differences in the volume, the sum of lengths squared, and the quality degrading for different values of φ: $\varphi = 0$, $\varphi = \pi/2$, and $\varphi = \arg\min \sqrt{A_\varphi^2 + B_\varphi^2}$. The orbit radius is 10,000 km, $K = 1000$ m, and $i = 60$ deg. To make explicit comparison between different cases possible J_2 harmonic is not included in the simulation here.

The presence of the J_2 perturbation also affects the formation degrading rate, although a careful choice of initial values could help to maintain the tetrahedron. Figures 18–20 depict the same values as the previous one, but with the presence of the J_2 perturbation force.

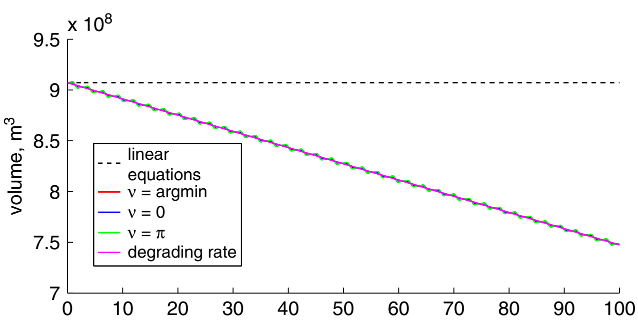


Fig. 15 Volume degrading.

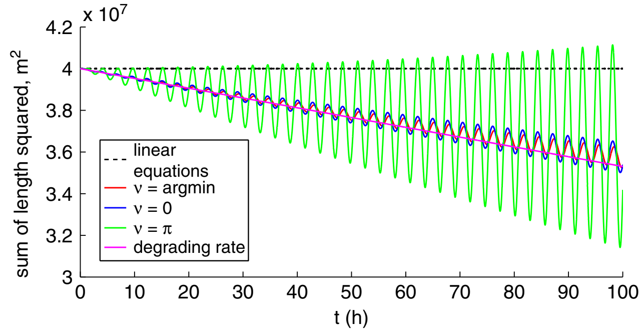


Fig. 16 Sum of lengths squared degrading.

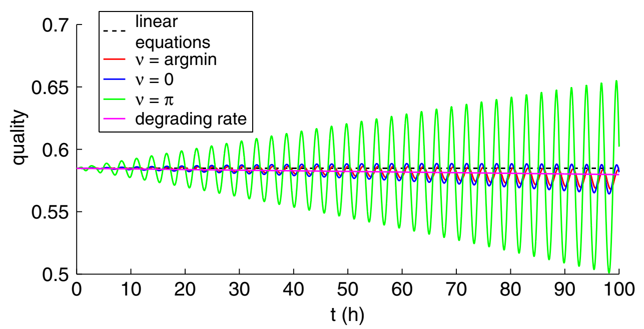


Fig. 17 Quality degrading.

B. Family 2

For the second family of solutions [Eq. (18)] using the same technique, we obtain the following expressions:

$$\mathbb{V}(\nu) = \frac{5\sqrt{6}}{4}K^3 - \varepsilon K^3\left(\frac{105\sqrt{6}}{32}\sin 3\varphi\right) - \varepsilon K^3 \frac{81\sqrt{15}}{8}\nu + \varepsilon K^3\left(\frac{201\sqrt{15}}{8}\sin\nu + \frac{15\sqrt{6}}{4}\sin(3\varphi+\nu) - \frac{45\sqrt{6}}{16}\sin(3\varphi+3\nu) + \frac{75\sqrt{6}}{32}\sin(3\varphi+2\nu)\right)$$

$$\mathbb{L}(\nu) = 90K^2 - \varepsilon K^2\left(\frac{315}{2}\sin 3\varphi\right) - \varepsilon K^2 243\sqrt{10}\nu + \varepsilon K^2\left(\frac{9}{2}\sin(3\varphi+2\nu) - 27\sin(3\varphi+3\nu) + 180\sin(3\varphi+\nu) + 243\sqrt{10}\sin\nu + 108\sqrt{10}\sin 2\nu\right) - \varepsilon K^2\nu(108\cos(3\varphi+\nu) + 216\sqrt{10}\cos\nu)$$

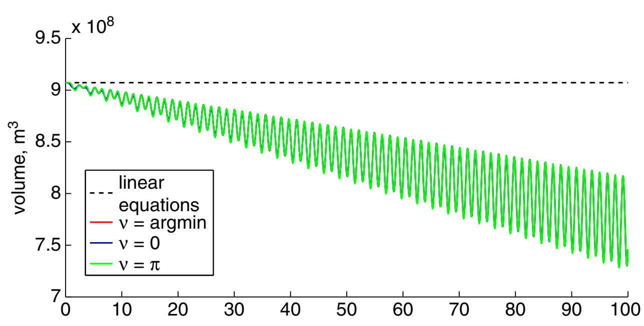


Fig. 18 Volume degrading.

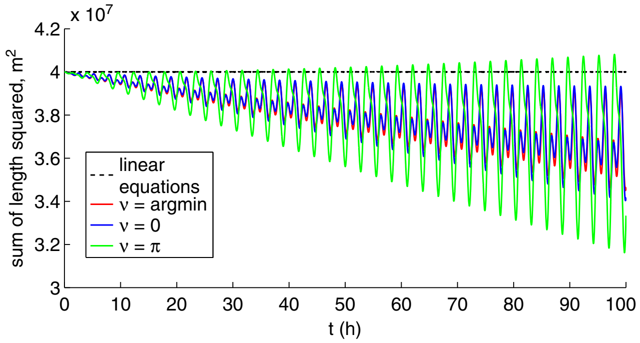


Fig. 19 Sum of lengths squared degrading.

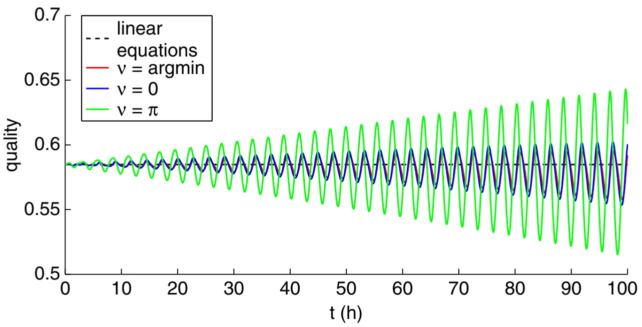


Fig. 20 Quality degrading.

$$\hat{\mathbb{V}}(\nu) = \frac{5\sqrt{6}}{4}K^3 - \varepsilon K^3 \frac{81\sqrt{15}}{8}\nu$$

$$\hat{\mathbb{L}}(\nu) = 90K^2 - K^2 \varepsilon \nu 243\sqrt{10}$$

The minimum oscillation amplitude is achieved when $\cos 3\varphi = -1$.

VI. Conclusions

In the paper, reference orbits were constructed for four satellites in a linear model for the tetrahedron to preserve the volume and quality according to the introduced criterion. In the case of a circular orbit, several families of initial conditions that ensured volume and quality preservation were found.

However, in a more precise motion model due to disturbances caused by nonlinear terms and J_2 perturbations, the tetrahedron degrades. It is shown that the degradation rate depends on tetrahedron size (which is expected) and the phase of the satellites, i.e., their initial placement at the reference orbit.

Taking into account disturbances during reference orbit construction can greatly reduce the degradation rate of the tetrahedron; however, it cannot eliminate it completely: after several weeks, the tetrahedron quality degrades too much. To achieve additional accuracy in a future problem investigation, it is proposed to include additional perturbations caused by eccentricity of the chief satellite and the J_2 harmonic, which can further prolong the mission.

Another possible improvement might be achieved using the curvilinear relative motion equations, which might better describe the full dynamics of the formation. In addition, some active control approaches (e.g., via atmospheric drag) might be investigated.

References

[1] Gill, E., Montenbruck, O., and D'Amico, S., "Autonomous Formation Flying for the PRISMA Mission," *Journal of Spacecraft and Rockets*, Vol. 44, No. 3, 2007, pp. 671–681.
[2] Kirschner, M., Massmann, F.-H., and Steinhoff, M., "GRACE," *Distributed Space Missions for Earth System Monitoring*, Springer, 2013, pp. 547–574.
[3] Griffith, J., Singh, L., and How, J., "Optimal Microsatellite Cluster Design for Space-Based Tracking Missions," *AIAA Guidance, Navigation and Control Conference and Exhibit*, AIAA Paper 2007-6543, 2007.
[4] Griffith, J. D., "Design and Control of Microsatellite Clusters for Tracking Missions," Ph.D. Thesis, Massachusetts Inst. of Technology, Cambridge, MA, 2007.
[5] Karlsson, T., Marklund, G. T., Figueiredo, S., Johansson, T., and Buchert, S., "Separating Spatial and Temporal Variations in Auroral Electric and Magnetic Fields by Cluster Multipoint Measurements," *Annales Geophysicae*, Vol. 22, No. 7, 2004, pp. 2463–2472.
[6] Sadeghi, S., and Emami, M., "Multi-Spacecraft Studies of the Auroral Acceleration Region: From Cluster to Nanosatellites," *Advances in Space Research*, Vol. 59, No. 5, 2017, pp. 1173–1188.
[7] Hametz, M., Conway, D., and Richon, K., "Design of a Formation of Earth Orbiting Satellites: The Auroral Lites Mission," *NASA Conference Publication*, NASA, 1999, pp. 295–308.
[8] Dow, J., Matussi, S., Dow, R. M., Schmidt, M., and Warhaut, M., "The Implementation of the Cluster II Constellation," *Acta Astronautica*, Vol. 54, No. 9, 2004, pp. 657–669.
[9] Guzmán, J. J., and Edery, A., "Mission Design for the MMS Tetrahedron Formation," *2004 IEEE Aerospace Conference Proceedings*, Vol. 1, IEEE Publ., Piscataway, NJ, 2004, p. 540.
[10] Guzman, J., and Schiff, C., "A Preliminary Study for a Tetrahedron Formation: Quality Factors and Visualization," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, AIAA Paper 2002-4637, 2002.
[11] Robert, P., Dunlop, M. W., Roux, A., and Chanteur, G., "Analysis Methods for Multi-Spacecraft Data," ESA Publications Division, The Netherlands, 1998, pp. 323–348.
[12] Daly, P. W., "The Tetrahedron Quality Factors of CSDS," Max Planck Inst. für Aeronomie Tech. Rept. MPAe–W–100–94–27, Katlenburg-Lindau, Germany, 1994, http://www.mpae.gwdg.de/~daly/tetra.html.
[13] Gim, D. W., and Alfriend, K. T., "Criteria for Best Configuration and Sub-Optimal Reconfiguration for MMS Mission," *Spaceflight Mechanics 2004*, edited by S. L. Coffey, A. P. Mazzoleni, K. K. Luu, and R. A. Glover, Vol. 119, 2004, pp. 947–968.
[14] Hughes, S. P., "Formation Tetrahedron Design for Phase I of the Magnetospheric Multiscale Mission," *Proceedings of the Flight Mechanics Symposium*, NASA Goddard Space Flight Center, Oct. 2003.
[15] Hughes, S. P., "Orbit Design for Phase I and II of the Magnetospheric Multiscale Mission," *Proceedings of the AAS 27th Rocky Mountain Guidance and Control Conference*, American Astronautical Soc., San Diego, CA, 2004.
[16] Hughes, S. P., "General Method for Optimal Guidance of Spacecraft Formations," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 2, 2008, pp. 414–423.
[17] Clemente, D. C., and Atkins, E. M., "Optimization of a Tetrahedral Satellite Formation," *Journal of Spacecraft and Rockets*, Vol. 42, No. 4, 2005, pp. 699–710.
[18] Roscoe, C. W., Vadali, S. R., and Alfriend, K. T., "Design of Satellite Formations in Orbits of High Eccentricity with Performance Constraints Specified over a Region of Interest," *Journal of the Astronautical Sciences*, Vol. 59, Nos. 1–2, 2012, pp. 145–164.
[19] Hughes, S., "Formation Design and Sensitivity Analysis for the Magnetospheric Multiscale Mission (MMS)," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, AIAA Paper 2008-7357, 2008.
[20] Roscoe, C. W. T., Vadali, S. R., Alfriend, K. T., and Desai, U. P., "Optimal Formation Design for Magnetospheric Multiscale Mission Using Differential Orbital Elements," *Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 4, 2011, pp. 1070–1080.
[21] Gong, S., Yunfeng, G., and Li, J., "Solar Sail Formation Flying on an Inclined Earth Orbit," *Acta Astronautica*, Vol. 68, Nos. 1–2, 2011, pp. 226–239.
[22] Parsay, K., and Schaub, H., "Designing Solar Sail Formations in Sun-Synchronous Orbits for Geomagnetic Tail Exploration," *Acta Astronautica*, Vol. 107, Feb.–March 2015, pp. 218–233.
[23] Parsay, K., Schaub, H., Schiff, C., and Williams, T., "Improving Magnetosphere in Situ Observations Using Solar Sails," *Advances in Space Research*, Vol. 61, No. 1, 2018, pp. 74–88.
[24] McInnes, C. R., MacDonald, M., Angelopolous, V., and Alexander, D., "GEOSAIL: Exploring the Geomagnetic Tail Using a Small Solar Sail," *Journal of Spacecraft and Rockets*, Vol. 38, No. 4, 2001, pp. 622–629.
[25] Schaub, H., and Alfriend, K. T., "J2 Invariant Relative Orbits for Spacecraft Formations," *Celestial Mechanics and Dynamical Astronomy*, Vol. 79, No. 2, 2001, pp. 77–95.
[26] Sabatini, M., Izzo, D., and Palmerini, G., "Minimum Control for Spacecraft Formations in a J2 Perturbed Environment," *Celestial Mechanics and Dynamical Astronomy*, Vol. 105, Nos. 1–3, 2009, Paper 141.
[27] Chernyshov, A., Chugunin, D., Mogilevsky, M., Moiseenko, I., Ilyasov, A., Vovchenko, V., Pulinets, S., Klimenko, M., Zakharenkova, I.,

Kostrov, A., et al., "Approaches to Studying the Multiscale Ionospheric Structure Using Nanosatellites," *Geomagnetism and Aeronomy*, Vol. 56, No. 1, 2016, pp. 72–79.
[28] Ermilov, N. O., "Reconstruction of Tetrahedra from Sets of Edge Lengths," *Mathematical Notes*, Vol. 91, Nos. 3–4, 2012, pp. 500–507.
[29] Wirth, K., and Dreiding, A. S., "Edge Lengths Determining Tetrahedrons," *Elemente der Mathematik*, Vol. 64, No. 4, 2009, pp. 160–170.
[30] Parthasarathy, V., Graichen, C., and Hathaway, A., "A Comparison of Tetrahedron Quality Measures," *Finite Elements in Analysis and Design*, Vol. 15, No. 3, 1994, pp. 255–261.
[31] Liu, A., and Joe, B., "On the Shape of Tetrahedra from Bisection," *Mathematics of Computation*, Vol. 63, No. 207, 1994, pp. 141–154.
[32] Vaddi, S. S., Vadali, S. R., and Alfriend, K. T., "Formation Flying: Accommodating Nonlinearity and Eccentricity Perturbations," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2, 2003, pp. 214–223.
[33] London, H. S., "Second Approximation to the Solution of the Rendezvous Equations," *AIAA Journal*, Vol. 1, No. 7, 1963, pp. 1691–1693.