Spatially inhomogenious modes of logistic equation with delay and small diffusion in a flat area

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We consider the boundary value problem from the population dynamics

$$\dot{N} = d\Delta N + r(1 - N_{t-1})N, \quad \left.\frac{\partial N}{\partial \nu}\right|_{\partial\Omega} = 0,$$

where $N = N(t,x) \in \mathbb{R}$ — population density; $N_{t-1} = N(t-1,x)$; $x \in \Omega \subset \mathbb{R}^2$; Δ — Laplace operator; D — diffusion coefficient; r — Malthusian coefficient of linear growth; ν — the direction of the outer normal to the border $\partial\Omega$ of bounded flat area Ω.

The objective is to detect and study the periodic and nonperiodic complex modes at $t \gg 1$.

Separately we consider the equation without diffusion:

$$\dot{N} = r(1 - N_{t-1})N.$$

In this case, it is well known that a unit equilibrium state $N \equiv 1$ becomes unstable when $r = \frac{\pi}{2}$.

If $r > \frac{\pi}{2}$ then $N \equiv 1$ is unstable and it is a T-periodic mode $N(t + T) \equiv N(t)$.

It is proving analytically when r is close to $\frac{\pi}{2}$, and numerically when $r > \frac{\pi}{2} + \varepsilon$.

Now we assume that $r > \frac{\pi}{2}$ or about $\frac{\pi}{2}$, and we consider the problem with diffusion:

$$\dot{N} = d\Delta N + r(1 - N_{t-1})N, \quad \left.\frac{\partial N}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{1}$$

Fix a parameter r. Then we have certainly a $N \equiv 1$ for $r < \frac{\pi}{2}$, and $N(t, x) = N(t)$ — periodic spatially homogeneous solution otherwise.

If the diffusion d is large enough ($d \gg 1$) then the spatially homogeneous solutions are certainly stable.

There is a critical diffusion d_{crit} for which this stability is lost.

In a numerical experiment we consider the area

$$\Omega = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}.$$

The area Ω is covered with a uniform grid with a step $h = 0.01$. The values in the appropriate squares of area Ω are considered identical and are denoted as $N_{i,j}$ ($i, j \in [1, M]$, where $M = 100$). Then the Laplace operator is replaced by its difference analogue

$$\Delta_n N_{i,j} = \frac{N_{i-1,j} - 2N_{i,j} + N_{i+1,j}}{h^2} + \frac{N_{i,j-1} - 2N_{i,j} + N_{i,j+1}}{h^2},$$

and the boundary value problem (1) is replaced by a system of differential-difference equations with the following boundary conditions:

$$N_{i,0} = N_{i,1}, N_{i,M} = N_{i,M+1}, \forall i \in [1, M],$$

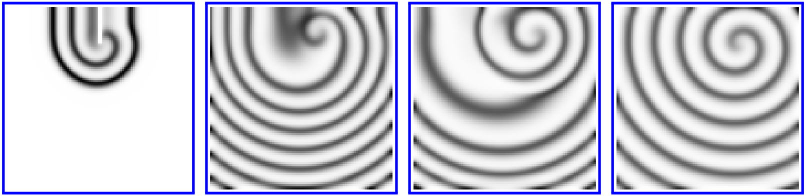
$$N_{0,j} = N_{1,j}, N_{M,j} = N_{M+1,j}, \forall j \in [1, M].$$

Thus, given the step $h = 0.01$, we consider the system of 10000 equations with delay. In the process of computing the value of d is varied.

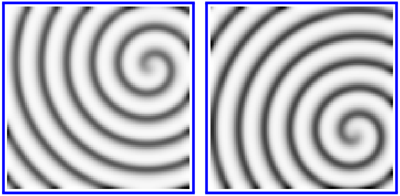
As a numerical method for solving the system with initial conditions $N_{i,j}(s) = \varphi_{i,j}(s), s \in [-1, 0]$, where $\varphi_{i,j}(s)$ are continuous by s functions, it was chosen the Dormand–Prince method of the fifth order with variable length of the integration step.

The calculations were performed on a computing cluster of YSU (МНИЛ «Дискретная и вычислительная геометрия» им. Б.Н. Делоне).

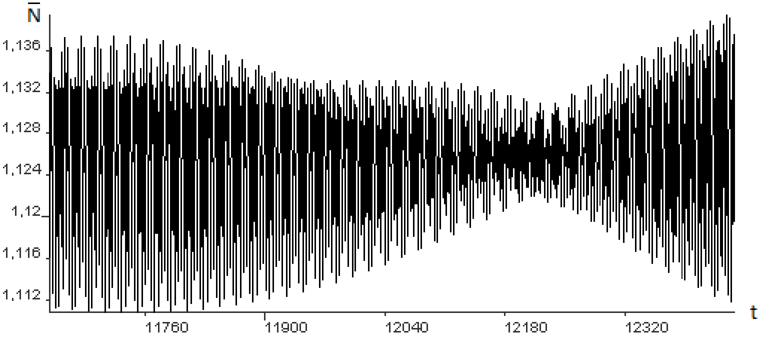
Spiral waves



Spiral wave generation by $r = 2$ and $d = 10^{-4}$. Times $t = 10, 48, 99, 753$.

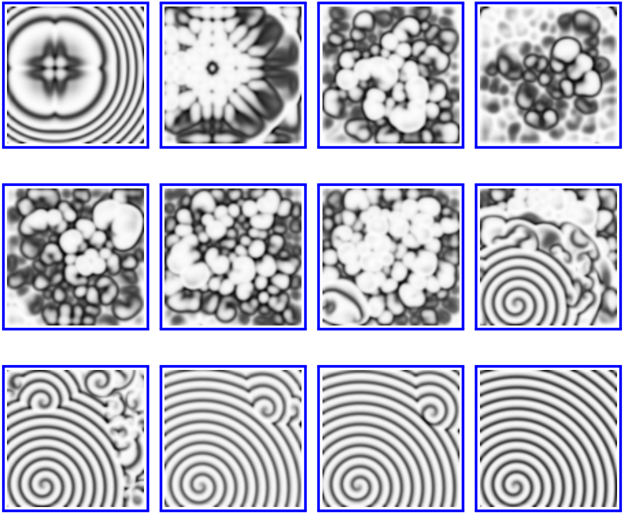


Wandering of the spiral wave. Times $t = 4713, 9807$.

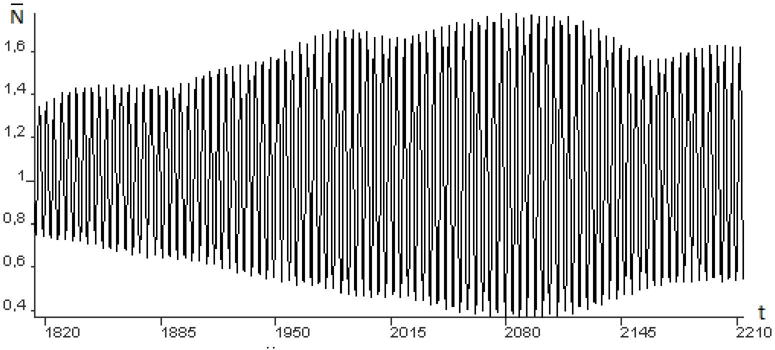


Distribution of the average value of $N_{i,j}$

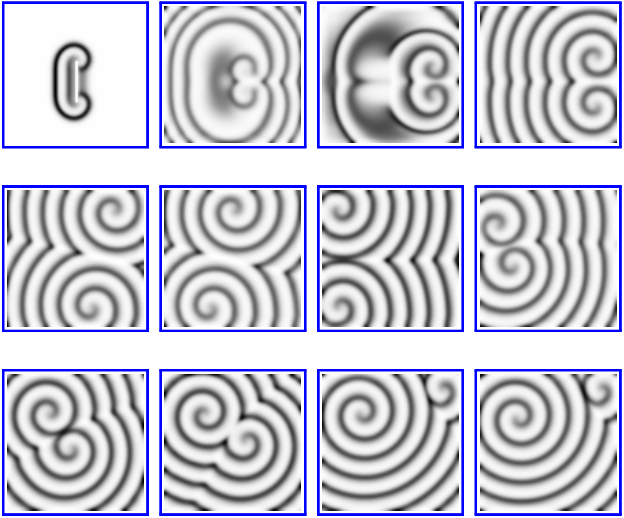
Spiral wave generation by $r = 3$ and $d = 5 * 10^{-5}$. Times $t = 5, 25, 83$.



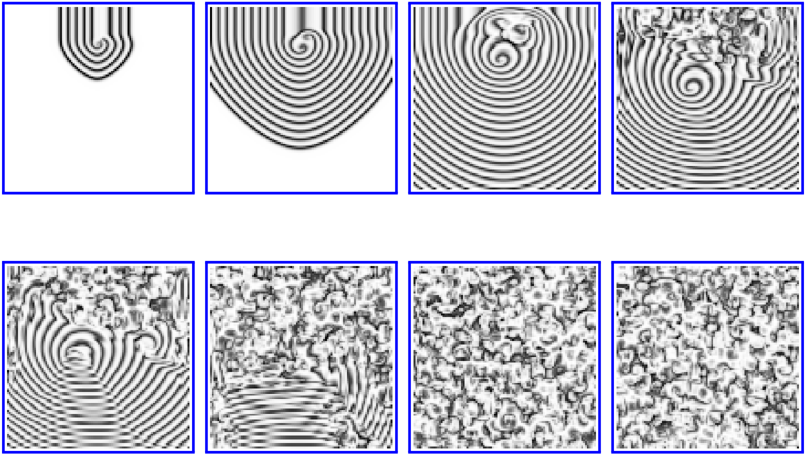
Spiral wave generation through spontaneous self-organization by $r = 2$ and $d = 3 * 10^{-5}$. Times $t = 120, 551, 2337, 5415, 6138, 8006, 8146, 8515, 8768, 9952, 10010, 11200.$



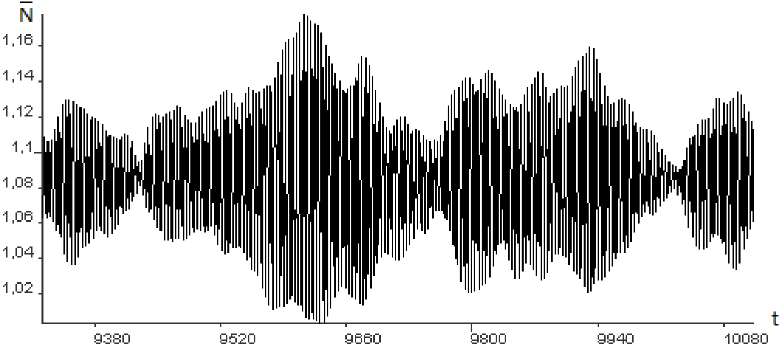
Distribution of the average value of $N_{i,j}$ in bubble structure



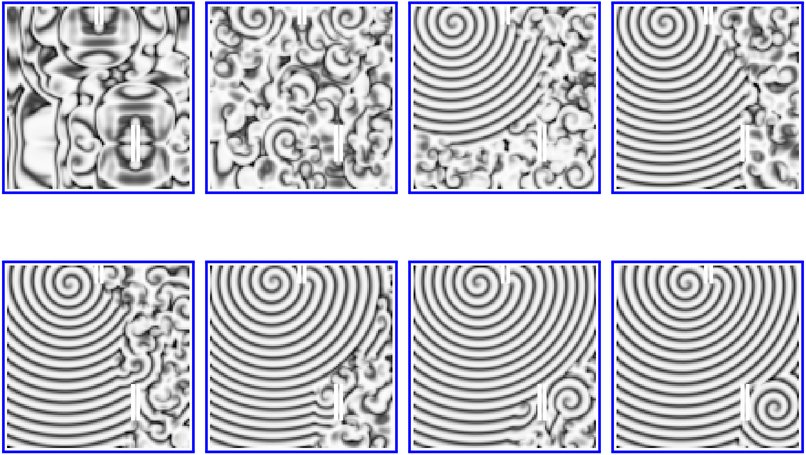
Double spiral wave generation by $r = 2$ and $d = 1 * 10^{-4}$. Times $t = 40, 57, 145, 600, 1600, 2200, 3400, 4880, 5442, 5771, 8173, 15000$.



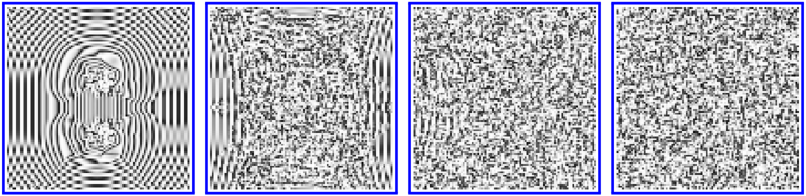
Chaotic structure by $r = 2$ and $d = 1 * 10^{-5}$. Times $t = 23, 51, 146, 374, 466, 750, 2161, 15000$.



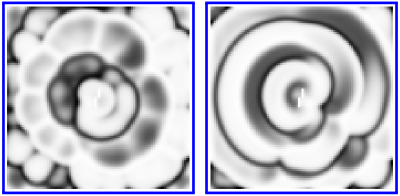
Distribution of the average value of $N_{i,j}$ in chaotic structure



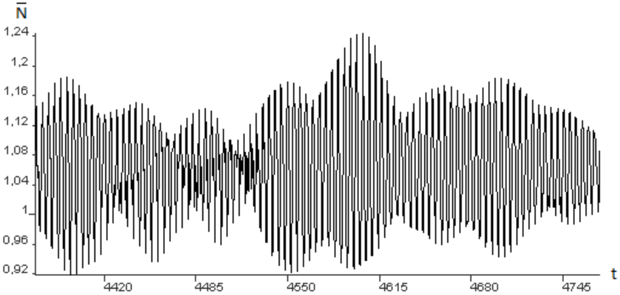
Coexistense of spiral wave and chaotic structure for a long time by $r = 2$ and $d = 1 * 10^{-5}$. Times $t = 410, 544, 2370, 4630, 7460, 8302, 9280, 9960$.



Numerical artifacts by $r = 2$ and $d = 5 * 10^{-6}$. Times $t = 202, 598, 917, 3691$.



Halo generation by $r = 2$ and $d = 5 * 10^{-5}$. Times $t = 2400, 4360$.



Distribution of the average value of $N_{i,j}$ in halo structure

Thank you for attention!