Self-excited Wave Processes in Chains of Unidirectionally Coupled Relaxation Oscillators

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Introduction

$$\dot{u} = \lambda[f(u(t-h)) - g(u(t-1))]u. \qquad (1)$$

$$u(t) > 0,\ \lambda \gg 1,\ h \in (0,1),\ f(u),\ g(u) \in C^1(\mathbb{R}_+),\ \mathbb{R}_+ = \{u \in \mathbb{R} : u \geq 0\},$$

$$f(0) = 1,\ g(0) = 0;\ \ f(u) = -a_0 + O(1/u),\ uf'(u) = O(1/u),$$
$$u^2 f''(u) = O(1/u),\ g(u) = b_0 + O(1/u),\ ug'(u) = O(1/u), \qquad (2)$$
$$u^2 g''(u) = O(1/u) \text{ as } u \to +\infty, \quad a_0 > 0,\ b_0 > 0.$$

$$\dot{u} = \lambda f(u(t-1))u, \tag{3}$$

$h = 1$

$$f(u) - g(u) \to f(u), \quad a_0 + b_0 \to a.$$

$$a > 1. \tag{4}$$

$$\dot{u}_j = d\,(u_{j+1} - u_j) + \lambda f(u_j(t-1))u_j, \quad j = 1, \ldots, m, \quad u_{m+1} = u_1, \tag{5}$$

$d = \text{const} > 0.$ $\lambda \gg 1.$

$$u_1 \equiv \ldots \equiv u_m = u_*(t, \lambda), \tag{6}$$

$u_*(t, \lambda)$

Main theorem

$$u_1 = \exp(x/\varepsilon), \quad u_j = \exp\left(x/\varepsilon + \sum_{k=1}^{j-1} y_k\right), \quad j = 2, \dots, m, \quad \varepsilon = 1/\lambda. \tag{7}$$

$$\dot{x} = \varepsilon d\,(\exp y_1 - 1) + F\big(x(t-1),\, \varepsilon\big), \tag{8}$$

$$\begin{gathered}\dot{y}_j = d\left[\exp y_{j+1} - \exp y_j\right] + G_j\big(x(t-1),\, y_1(t-1),\, \dots,\, y_j(t-1),\, \varepsilon\big), \\ j = 1, \dots, m-1,\end{gathered} \tag{9}$$

$$y_m = -y_1 - y_2 - \dots - y_{m-1},\ F(x, \varepsilon) = f\big(\exp(x/\varepsilon)\big),$$

$$G_j(x, y_1, \dots, y_j, \varepsilon) = \frac{1}{\varepsilon}\Big\{f\Big(\exp\Big(x/\varepsilon + \sum_{k=1}^{j} y_k\Big)\Big) - f\Big(\exp\Big(x/\varepsilon + \sum_{k=1}^{j-1} y_k\Big)\Big)\Big\},$$
$$j = 1, \dots, m-1.$$

$0 < \sigma_0 < a - 1$, $\mathscr{F}$ – Banach space of functions $\varphi(t) = \big(\varphi_1(t), \ldots, \varphi_m(t)\big)$ on $-1 - \sigma_0 \le t \le -\sigma_0$.

$$||\varphi||_{\mathscr{F}} = \max_{1\le j\le m} \Big(\max_{-1-\sigma_0\le t\le -\sigma_0} |\varphi_j(t)| \Big). \tag{10}$$

$$S = \big\{\varphi(t) = \big(\varphi_1(t), \ldots, \varphi_m(t)\big): \ \ \varphi_1 \in S_1,\, \varphi_2 \in S_2, \ldots, \varphi_m \in S_m\big\} \subset \mathscr{F}.$$

$S_1 = \{\varphi_1(t) \in C[-1-\sigma_0, -\sigma_0] | -q_1 \le \varphi_1(t) \le -q_2, \quad \varphi_1(-\sigma_0) = -\sigma_0\}$, $q_1 > \sigma_0$, $q_2 \in (0, \sigma_0)$, $S_2, \ldots, S_m \subset C[-1-\sigma_0, -\sigma_0]$.

$$\big(x_\varphi(t,\, \varepsilon),\, y_{1,\varphi}(t,\, \varepsilon),\, \ldots,\, y_{m-1,\varphi}(t,\, \varepsilon)\big), \quad t \ge -\sigma_0 \tag{11}$$

$\Pi_\varepsilon : S \to \mathscr{F}$

$$\begin{gathered}\Pi_\varepsilon(\varphi) = \big(x_\varphi(t + T_\varphi,\, \varepsilon),\, y_{1,\varphi}(t + T_\varphi,\, \varepsilon),\, \ldots,\, y_{m-1,\varphi}(t + T_\varphi,\, \varepsilon)\big), \\ -1 - \sigma_0 \le t \le -\sigma_0.\end{gathered} \tag{12}$$

$$\Pi_0(\varphi) = \big(x_0(t), y_1^0(t+T_0, z), \ldots, y_{m-1}^0(t+T_0, z)\big)\Big|_{z=(\varphi_2(-\sigma_0),\ldots,\varphi_m(-\sigma_0))},$$
$$-1-\sigma_0 \le t \le -\sigma_0. \tag{13}$$

$$x_0(t) = \begin{cases} t & \text{if } 0 \le t \le 1, \\ 1 - a(t-1) & \text{if } 1 \le t \le t_0 + 1, \\ -a + t - t_0 - 1 & \text{if } t_0 + 1 \le t \le T_0, \end{cases} \qquad x_0(t+T_0) \equiv x_0(t). \tag{14}$$

$$\dot{y}_j = d\left[\exp y_{j+1} - \exp y_j\right] \quad y_j(1+0) = y_j(1-0) - (1+a)\,y_j(0),$$
$$y_j(t_0+1+0) = y_j(t_0+1-0) - (1+1/a)\,y_j(t_0), \quad j = 1,\ldots,m-1, \tag{15}$$
$$y_m = -y_1 - y_2 - \ldots - y_{m-1},$$

$$(y_1,\ldots,y_{m-1})\big|_{t=-\sigma_0} = (z_1,\ldots,z_{m-1}), \tag{16}$$

$t_0 = 1 + 1/a.$

Theorem (on $\mathbf{C}^1$-convergence)

There exist small enough $\varepsilon_0 = \varepsilon_0(S) > 0$ *such that for all* $0 < \varepsilon \le \varepsilon_0$ *the operator* Π_ε *are defined on* S *and*

$$\lim_{\varepsilon\to 0}\ \sup_{\varphi\in S} ||\Pi_\varepsilon(\varphi) - \Pi_0(\varphi)||_{\mathscr{F}} = 0,$$
$$\lim_{\varepsilon\to 0}\ \sup_{\varphi\in S} ||\partial_\varphi\Pi_\varepsilon(\varphi) - \partial_\varphi\Pi_0(\varphi)||_{\mathscr{F}_0\to\mathscr{F}_0} = 0. \tag{17}$$

$$z \to \Phi(z) \stackrel{\text{def}}{=} \left(y_1^0(t,z), y_2^0(t,z), \ldots, y_{m-1}^0(t,z)\right)\Big|_{t=T_0-\sigma_0}, \tag{18}$$

$z = (\varphi_2(-\sigma_0), \ldots, \varphi_m(-\sigma_0))$.

$z = z_*$

$$\varphi_*(t) = \big(\varphi_1^*(t), \ldots, \varphi_m^*(t)\big): \ \varphi_1^*(t) = x_0(t),\ \varphi_j^*(t) = y_{j-1}^0(t+T_0, z_*),$$
$$j = 2, \ldots, m, \quad -1-\sigma_0 \le t \le -\sigma_0$$

Theorem (Compliance Theorem)

For any fixed point $z = z_*$ *of map* $\Phi(z)$ (18), *such that* $\det\left(I - \Phi'(z_*)\right) \neq 0$, *there exist relaxation cycle of system* (8), (9)*. This cycle exists for all small enough* $\varepsilon > 0$ *and is exponentially orbitally stable (unstable) if* $r_* < 1$ (> 1), *where* r_* *– spectral radius of matrix* $\Phi'(z_*)$*.*

$$a > m - 1. \tag{19}$$

$$z_j = -\frac{1}{a}\,\ln\frac{1}{d} + v_j, \quad j = 1, \dots, m-1, \tag{20}$$

$d \to 0$

$$y_j(t,v,d) = -\frac{1}{a}\,\ln\frac{1}{d} + v_j + O(d^{1-(m-1)/a}) \quad \text{if} \quad 0 \le t < 1, \tag{21}$$

$$y_j(t,v,d) = \ln\frac{1}{d} + \omega_j^0(t,v) + O(d^{1-(m-1)/a}) \quad \text{if} \quad 1 \le t < t_0 + 1, \tag{22}$$

$$y_j(t,v,d) = -\frac{1}{a}\,\ln\frac{1}{d} + \psi_j(v) + O(d^{1-(m-1)/a}) \quad \text{if} \quad t_0 + 1 \le t \le T_0, \tag{23}$$

$$\dot{\omega}_j = \exp\omega_{j+1} - \exp\omega_j, \quad j = 1, \ldots, m-2,$$
$$\dot{\omega}_{m-1} = -\exp\omega_{m-1},$$
$$\omega_j\Big|_{t=1} = -a\,v_j, \quad j = 1, \ldots, m-1$$
$$\omega^0_{m-1}(t, v) + \ldots + \omega^0_{m-s}(t, v) =$$
$$= -\ln\left\{\frac{(t-1)^s}{s!} + \sum_{\ell=0}^{s-1}\frac{(t-1)^\ell}{\ell!}\exp\left(a\sum_{j=1}^{s-\ell}v_{m-j}\right)\right\}, \tag{24}$$
$$s = 1, \ldots, m-1.$$

$$\psi_j(v) = \omega_j^0(t,v)\big|_{t=t_0+1} - (1+1/a)\omega_j^0(t,v)\big|_{t=t_0}, \quad j=1,\dots,m-1. \tag{25}$$

$$v_j \to \psi_j(v), \quad j=1,\dots,m-1. \tag{26}$$

$$\alpha_s = -v_{m-1} - \ldots - v_{m-s}, \quad s = 1, \ldots, m-1$$

$$\alpha_k \to \ln\big(r_{1,k} + \exp(-a\alpha_k)\big) - (1 + 1/a)\ln\big(r_{2,k} + \exp(-a\alpha_k)\big), \\ k = 1, \ldots, m-1, \tag{27}$$

where

$$r_{1,1} = 1 + 1/a, \quad r_{2,1} = 1/a, \tag{28}$$

$$r_{1,k}(\alpha_1, \ldots, \alpha_{k-1}) = \frac{(1+1/a)^k}{k!} + \sum_{\ell=1}^{k-1} \frac{(1+1/a)^\ell}{\ell!} \exp(-a\alpha_{k-\ell}),$$

$$r_{2,k}(\alpha_1, \ldots, \alpha_{k-1}) = \frac{1}{a^k\, k!} + \sum_{\ell=1}^{k-1} \frac{1}{a^\ell\, \ell!} \exp(-a\alpha_{k-\ell}), \quad k = 2, \ldots, m-1. \tag{29}$$

$(\alpha_1^*, \ldots, \alpha_{m-1}^*)$,

$$z_* = (z_1^*, \ldots, z_{m-1}^*), \quad z_j^* = -\frac{1}{a} \ln \frac{1}{d} + v_j^* + O(d^{1-(m-1)/a}), \tag{30}$$
$$j = 1, \ldots, m-1, \quad d \to 0,$$

where $v_{m-1}^* = -\alpha_1^*$, $v_{m-s}^* = \alpha_{s-1}^* - \alpha_s^*$, $s = 2, \ldots, m-1$.

Thank you for attention!